Word Problems in Russia and America

Extended version of a talk at the Meeting
of the Swedish Mathematical Society. June, 2005

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1. Introduction

I am a mathematician of Russian origin interested in mathematical education. I have done research and taught for two decades in Russia, a decade in USA and six years in Brazil. All this time I was interested in and contributed to mathematical education.

I left Russia forever not without reason. Russia had and still has some ugly features with which I don't want to have anything common. However, Russian public mathematical education is better than American and Brazilian and it has been so for several decades including the Communist period. This is a paradox, but it is not created by me. Life itself created this paradox and we need to study it. The contrast between a very productive usage of word problems in Russia and a failure to use them in USA is one of the most outstanding aspects of this paradox.

This text is focused on word problems and their place in education. In particular, I want to compare the very successful usage of word problems in Russia and contradictory, inefficient and often immature treatment of word problems in USA.

In this text I refer to many sources. I must warn the reader that I had absolutely no possibility to describe each author's outlook. My quotes only direct you to the sources, but cannot substitute a thorough study of them. Some of these sources are so excellent that I had to suppress the desire to quote them in toto. Still I could not deny myself the pleasure of including two articles into my text; they are put in the appendix.

It is a truism that mathematics is an abstract science. It deals with abstractions which cannot be seen, heard, touched or smelled. Special terms are used to describe these abstractions and one might think that handling these terms properly is all we need to formulate and solve any mathematical problem. But the practice of teaching and research is different. When we discuss research ideas, we wave our hands, draw pictures and use words, which have no exact meaning. When we teach mathematics, we often give students so-called “word problems”, which use various non-mathematical words. Does it make sense? And if it does, why? These are the main questions we address in this article.

First of all let me make several assumptions which I shall call “axioms” because I shall not prove them.

Axiom 1. Mathematics already plays an important role in modern civilization and this role will grow.

Axiom 2. Mathematical education already plays an important role in modern civilization and this role will grow.

Axiom 3. For their existence and functioning mathematics and mathematical education need special institutions and conditions. Children's competence in mathematics can not spontaneously grow out of their everyday lives or social activities.

Axiom 4. Mathematics is so difficult that it can not be done with just one mental function, for example abstract or formal thinking. It needs collaboration of all mental functions including visualization and other forms of imagination.

Although I shall not prove these “axioms”, much of this text serves as illustration and detalization of them.

Since this text is about word problems, we need first of all to define the subject. To keep as close as possible to the exact meaning of the words, I suggest that a non-word problem is a problem, which is formulated using only mathematical symbols and technical words like “Solve the equation...” Correspondingly, a word problem is a problem which uses non-mathematical words. Being put into a mathematical context, these words need to be interpreted mathematically and this greatly contributes to word problems' worth, thrill and perils.

Since this text is not a detective story, I shall not keep in secrecy my main thesis. Here it is:

Main Thesis. Word problems are very valuable in teaching mathematics not only to master mathematics, but also for general development. Especially valuable are word problems solved with minimal scolarship, without algebra, even sometimes without arithmetics, just by plain common sense. The more naive and ingenuous is solution, the more it provides the child's contact with abstract reality and independence from authority, the more independent and creative thinker the child becomes. to this idea I was inspired by Barry Mazur's excellent article. Here is a quote from it:

> We are surrounded by experts. Often we cannot live without them. $< \cdots >$How difficult it is, nowadays, to think about anything without relying on some external authority.
>
> But independence from authority still lies at the core of a few modes of thought. Encounters with art and encounters with mathematics – even the simplest unscary math – can be exhilarating, for that reason: In experiencing the impact of a work of art, or understanding a piece of mathematics, you are – or at least you can be – entirely on your own, with no authority in sight. $< \cdots >$
>
> When, for instance, we consider some lines of poetry – like Theodore Roethke's "Snail, snail glister me forward / Bird, soft-sigh me home, / Worm, be with me. / This is my hard time." – authorities of various stripes can bring us closer to them, but in the end, the gurus, the learned scholars, the biographers and the grammarians must get out of our way as we experience these lines.
>
> Art and mathematics invite us to leave our authoritative guides behind and ultimately to do the essential work on our own. [**Mazur**]

Study of mathematics in school is useful because it teaches children to understand complex, rigorous or abstract matters. When we teach children to solve problems in school, we do not expect them to meet exactly and literally the same problems in later life. Mathematical education would be next to useless if its only use were literal. We want much more, we want to teach children to solve problems in general. In this respect traditional word problems are especially valuable, because to solve a word problem, you have to *understand* what is said there. This function of word problems is very poorly understood in America.

Although American educators pay lip service to the memory of George Polya, they often neglect his opinions. George Polya attached special importance to solving word problems in school. He wrote:

> **Why word problems?** I hope that I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary schools is to teach the setting up of equations to solve word problems. Yet there is a strong argument in favor of this opinion. In solving a word problem by setting up equations, the student **translates** a real situation into mathematical terms; he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out. [**Polya**], p. 59

When I read [**Polya**] in Russia, I thought that it was just a good book. Now I understand how polemical it actually was. Its first two chapters are devoted to two Cinderellas of American education: classical geometry and word problems.

At the K-12 level there is not much room for sophisticated formalisms of professional mathematics, so non-word problems, which deal with formalisms, are exercises, which are necessary, but not exciting. No wonder that most interesting problems available at this level are word problems. They bring to the classroom a plethora of images, such as coins, buttons, matches and nuts, time and age, work and rate, distance and speed, length, width, height, perimeter, area and volume, fields, boxes, barrels, balls and planets, price, percentage, interest and discount, mass and mixture, ships and current, planes and wind, pumps and pools etc. etc. It is an invaluable experience for children to discern those formal characteristics of these images, which should be taken into account to solve the problem.

The world of word problems is enormously diverse, so there are word problems for all ages. The youngest children need some real, tangible tokens, which often are called *manipulatives.* That is why coin problems are so appropriate in elementary school. American educators enormously exaggerate importance of manipulatives in the literal sense, but don't know what to do with older children. In a few years children's imagination develops so that they can use *imaginary* or *mental manipulatives.* In [**Toom.Man**] I suggested that the main educative value of word problems is that they serve as mental manipulatives, paving children's road to abstract thinking.

Pumps and other mechanical appliances are easy to imagine working at a constant rate. Problems involving rate and speed should be (and in Russia are) common already in middle school. Trains, cars and ships are so widely used in textbooks not because all students are expected to go into transportation business, but for another, much more sound reason: these objects are easy to imagine moving at constant speeds and because of this are appropriate as reifications of the idea of uniform movement, which, in its turn, can serve as a reification of linear function. Thus, we can move children further and further on the way of *dereification*, that is development of abstract thinking.

What is at least equally important, in my opinion, is that solving word problems, children have to comprehend and translate into mathematics a multitude of verbs, adverbs and syntactic words indicating actions and *relations* betwen objects, such as put, give, take, bring, fill, drain, move, meet, overtake, more, less, later, earlier, before, after, from, to, between, against, away etc. Although I say “children”, I actually mean a wide range of ages, including college undergraduates, for whom all this may be quite a challenge [**Toom.How**].

However much I like Polya’s statement quoted above, it needs some corrections. First, usually it is practically impossible to bring real objects, such as ships or planes, into classroom. So that “situation”, which a student translates, is not really real; it is a situation described in words. and not just in words, but in a meaninful text, whose meaning is imbedded into certain syntactic constructions. Second, some word problems can and should be solved arithmetically, without algebra. The case of arithmetical word problems is even more challenging for an educational theoretician, because in this case it is even more evident that we face a question: into what does a student translates those words? Into numbers and nothing else? And in the case of algebraic word problems – do we translate their wordings into equations and nothing else? I believe that in both cases the answer is negative: beyond numbers and equations there is something else, which we have to keep in mind to solve problems correctly.

An analogous question arises about the wording: what does that student translate: only isolated words? Evidently, not, because if we were given just a list of isolated words without their order, we would fail or it would be a difficult puzzle to sort them out.

In my opinion, all these questions are very interesting and important and I by no means pretend to answer them completely. I only ask the reader to keep these questions in mind, because much (perhaps, all) of what is written below is relevant to these questions and helps to clarify or at least illustrate them.

Part I. Word Problems in Russia

2. Arithmetics

In Russia presence, even abundance of word problems in mathematical education was always normal. “Always” does not mean only Soviet time; it was so already in the 19-th century. I don’t even remember any special term for word problems from my childhood because the word “problem” usually meant a word problem, while non-word problems were called “exercises”. For example, the title of the book by Berezanskaya [**Berez**], which was used in 1930s-1940s, mentions “problems and exercices in arithmetics for 5 and 6 grades.” Here “problems” mean word problems, which constitute a majority of the book’s 2354 assignments.

Now Russians speak about “text problems”,which means the same as “word problems”. Traditionally, in Russia word problems are not considered only a part of algebra. First, they always were present much earlier, from the very beginning of elementary school, when they were and still are supposed to be solved without algebra. Second, they provide contacts of algebra with geometry and physics and generally the world of material objects.

In my childhood elementary school children were required (and I hope still are) to solve arithmetical word problems writing a series of questions, answering them by computations and finally write an explicit answer. This is an example, a problem from a Russian problem book for the 4-th grade. (I enumerate the problems included in this text for reference purposes.)

Problem 1 A library needs to bind 4500 books. One shop can bind these books in 30 days, another shop can do it in 45 days. How many days are necessary to bind all the books if both shops work at the same time? [**Moro.4.2**], p. 73 $\boxed{\textbf{P : books}}$

A model solution may look as follows:

How many books can the 1-st shop bind in 1 day?

4500/30 = 150

How many books can the 2-d shop bind in 1 day?

4500/45 = 100

How many books can the two shop bind in 1 day?

150+100=250

In how many days can the two shops bind the books?

4500/250=18

Answer: the two shops can bind the books in 18 days.

Problem 1 $\boxed{P: books}$ may be called a "forward problem" because it can be solved in a straightforward way, just performing several arithmetical operations with evident meanings. However simple, forward problems are an indispensable stage of development of every child's mathematical competence. This stage, in its turn, consists of several stages with growing number of steps in their solutions and every child should pass this ladded step by step. For example, the problem 1 $\boxed{P: books}$ is four-step since its solution includes four arithmetical opertions. It should be given to children who have already solved problems with smaller number of steps.

The following problem is from Singapore school textbook for the 6-th grade:

Problem 2 A motorist travelled from Town A to Town B. He took 2 hours to cover the first 1/2 of the journey at an average speed of 75 km/h. If his average speed for the whole journey was 60 km/h, find his average speed for the second 1/2 of the journey. [**Sing.6A**], p. 90 $\boxed{\mathbf{P: journey}}$

This problem is also forward. It can be solved in five steps as follows.

Half of the journey was $2 \text{ h} \times 75 \text{ km/h} = 150 \text{ km}$.
The total journey was $2 \times 150\ km = 300$ km.
The total time was $300 \text{ km} \div 60 \text{ km/h} = 5 \text{ h}$.
The time spent on the second half was $5 \text{ h} - 2 \text{ h} = 3 \text{ h}$.
The average speed on the second half was $150 \text{ km} \div 3 \text{ h} = 50 \text{ km/h}$.

Thus the Russian and Singapore programs go roughly hand in hand in style and difficulty of arithmetical problems. Probably, both countries found a good trade-off between different requirements: their programs are productive enough and at the same time understandable for an average student and an average teacher.

Let me emphasize that to solve a four or five-step problem is more difficult and more useful for a child than to solve four or five one-step problems. The greater is the number of operations, the more difficult it becomes to choose, which operation to perform at every step and with which numbers. This needs *planning* of actions, which is much more complicated than with one-step problems. To explain this idea better, let us compare solving word problems with solving chess problems. There, to solve a problem "The whites begin and win in five moves", one has to deal with sequences of nine half-moves (five moves of whites and four moves of blacks). But the number of such sequences grows roughly exponentially as a function of the number of half-moves and very soon becomes too large even for moderm computers. In fact, humans do not consider all such sequences; they do something less boring and more creative. The same takes place when we solve multi-step word problems: we do not try all possible sequences of operations with all possible numbers. What do we do instead of it?

Several ideas are helpful in this respect. Right now let us concentrate on one of them: the idea of dimension. For example, in movement problems there are three most used kinds of quantities – *distances*, *times* and *velocities* and it does not make sense to add quantities with different dimensions or to multiply distance and time. The problem

1 $\boxed{P: books}$ belongs to the realm of so-called "work problems" where also there are three kinds of quantities: *work*, *time* and *rate*, the latter's dimension being work divided by time. The key idea in work problems is that rate is additive. Thus solving word problems helps children to comprehend dimension and physical quantities, sometimes called concrete numbers. The phrase *concrete numbers* usually means numbers with units, for example three apples, ten meters, 6 dollars 99 cents, 2 hours 30 minutes, 220 volts and a lot of others. In my opinion, the quantities are really fundamental while concrete numbers are just their representations. For example, 1 meter and 100 centimeters represent one and the same quantity in different ways. I am sure that quantities and concrete numbers are very important and that it is essential for children to study them. You can find more on this at Michel Delord's web site [**Delord**].

Russian children solve plenty of word problems at all school levels. Their difficulty continually grows from one grade to another. Roughly, Russian children start to solve one-step word problems at the end of the first grade, then they start to solve two-step word problems at the end of the second grade, then they start to solve three-step word problems at the end of the third grade, and then they start to solve four-step word problems at the end of the fourth grade. All this time they solve problems by arithmetical means, without algebra. This training provides foundation for solving more sophisticated problems in the following grades. In the fifth grade, when children are quite comfortable with many word problems, algebra comes easily and allows to solve many more problems. I made these conclusions studying Russian school textbooks. I chose concervative ones including those written by M. I. Moro and others [**Moro.4.1, Moro.4.2**]. (To spare children from carrying too heavy load, every team of authors prepares two textbooks for each year: first half-year and second half-year.)

A more ambitious program is led by B. P. Geidman. It is used in a minority of schools. Here are a few problems from his textbooks for the second grade:

Problem 3 Vintik and Shpuntik agreed to go to the fifth car of a train. However, Vintik went to the fifth car from the beginning, but Shpuntik went to the fifth car from the end. How many cars the train needs to have for the two friends to get to one and the same car? [**Geidman.2.1**], p. 9

Problem 4 Igor and his two friends played chess. Everyone played two games. How many games were played? [**Geidman.2.1**], p. 73

Problem 5 All the numbers from 1 to 99 were written one after another. How many times the digit 5 was written? [**Geidman.2.2**], p. 63

Problem 6 Three friends study in the first, second and third grades. Their family names are Ivanov, Petrov and Semyonov. The youngest of them has no siblings. Semyonov studies in one group with Petrov's sister, he is the oldest of the three friends. Name the first-grader, second-grader and third-grader. [**Geidman.2.2**], p. 86

Problem 7 Dasha and Masha have as many candies as Kolya and Tolya. Masha has 5 candies, Kolya has 8 candies. Who has more candies, Dasha or Tolya? [**Geidman.2.2**], p. 107

I think that it is already visible that these books contain interesting, but appropriate problems. Under favorable conditions, this program may be very productive.

Although modern Russian textbooks share the main advantage of textbooks of Communist period –abundance of word problems of appropriate and continuosly growing difficulty, they look better now. The pictures are more bright (and always relevant) and instead of Soviet propaganda, they are full of most popular characters of children's literature, often fantastic. These are two examples:

Problem 8 Two witches argued, what is faster: mortar or broomstick. They flied one and the same distance of 288 km. The witch in mortar made it in 4 h and the witch on broomstick made it in 3 h. What is greater, speed of mortar or speed of broomstick and how much? [**Geidman.4.1**], p. 60. **P : witches**

The Western witches typically fly on broomsticks, but Russian witches use mortars as well. They are not so fast, but more comfortable, especially in winter.

Problem 9 When Ivan Tsarevich came to the Magic Kingdom, Koschey was as old as Baba Yaga and Ivan Tsarevich together. How old was Ivan Tsarevich when Koschey was as old as Baba Yaga was when Ivan Tsarevich came to the Magic Kingdom? [**Geidman.4.1**], p. 104. **P : Ivan**

Ivan Tsarevich, Koschey and Baba Yaga are well-known characters of Russian folk tales. I think that the problem 9 *P : Ivan* is a piece of art: none number is given, but it is solvable!

Russian educators are not infallible and make mistakes. For example, some people proposed not to have children solve arithmetical word problems because they can be easily solved using algebra. Several school books were written in this vein. This idea proved a mistake and several prominent Russian educators attacked it. B. P. Geidman, author of textbooks quoted above, lately wrote in his theses "On school mathematics education": "To preserve arithmetics" [**Geidman.Theses**]. Another Russian educator, L. D. Kudriavtsev, lately wrote in more detail:

> There is a point of view that it does not make sense to pay much attention to the solving of arithmetical word problems because in the future they will be solved in a simpler way, by making algebraic equations. Realization of this point of view has led to decreasing of the number of hours devoted to study

> of arithmetics in the school curriculum, which quickly resulted in decreasing of the level of logical thinking of students. This is connected with the fact that the main goal of solving arithmetical word problems is development of choldren's thinking, their ability to make correct logical conclusions from analysis of data given in the problem and to use them to solve the problem. The experience collected in the preceding decades has shown that the method of developing of children's logical thinking by solving arithmetical problems at a certain age has proved itself completely sound, so it seems very unwise to abandon it. It is necessary to add that nobody has yet found or suggested another efficient method to develop children's logical thinking. [**Kudr**], p. 55

To safeguard arithmetics, the following phrase was included in the Russian Federal Standards:

> Solving word problems by arithmetical means (using schemata, tables, short records and other models). [**Rus.Fed**], p. 41

However, the worst possible result of elimination of arithmeticl word problems, even if it prevailed, would be sliding down to a situation, which still remains a piece of cake by comparison with many American elementary and middle public schools, where multi-step word problems are not solved at all, with or without algebra.

Alexander Shevkin, a teacher and an author, is one of the prominent modern Russian educators. One of his books [**Shevkin.G**] contains 275 problems and many exlanations. Shevkin includes many historical problems, thereby supporting historical continuity, which is very important in my opinion. Among others, he includes problems by Magnitsky (Russia, 17-18 century), Kiselev (Russia 19-20 century), Polya (Hungary-USA, 20 century), Newton (England, 17-18 century), Euler (Switzerland-Russia, 18 century), Fibonacci (Italy, 12-13 century), Zu Chongzhi (China, 5 century), Adam Ries (Germany, 15-16 century), Alquin (England-France, 8-9 century), Perelman (Russia, 20 century), Euclid (Alexandria, 4-3 century B.C.), Al-Kashi (Iran-Uzbekistan, 14-15 century), Bézout (France, 18 century), Yevtushevsky (Russia, 19 century), Smullian (USA, 20 century), unknown authors (China, 1-2 centuries). I believe that this rich sample inspires teachers, helps them to feel that they continue a great tradition and their self-esteem helps their students to work hard and believe that it is worth while.

In Russia sound opinions often come from teachers and researchers alike. It is even difficult to draw a line between them. Vladimir Arnold, famous for his research in mathematics, always paid much attention to education and repeatedly emphasized importance of history, traditions and traditional style word problems.

3. Find two numbers given their sum and difference

When I was in the fourth grade (ten years old), we solved the following problem (I don't remember exact numbers):

Problem 10 A plane has two gasoline tanks. The total amount of gasoline in both tanks is 24 liters. The first tank contains 4 liters of gasoline more than the second. How much gasoline is there in each tank?

This problem is not exactly forward because it is not evident what to do first. The teacher stated the problem and invited us to think about it. She also drew a picture on the board showing two similar vessels, but the level of liquid in first one was higher than in the second. Then she drew a dotted line in the second vessel at the same height as the level of liquid in the first vessel. It was clear to me that the first operation should be to add the two given numbers. (It is equally good to subtract 4 from 24, but let us ignore this.) But which question to ask so that this operation of addition would answer it? I suggested to ask: “How much gasoline is there in two first tanks?” For me this question was perfectly clear: I meant two copies of one and the same thing. However, the teacher did not understand my suggestion. She suggested: “How much gasoline would be in the two tanks if the second tank contained as much gasoline as the first one?” It was clear from her self-assurance that this suggestion was recommended by some guidebook for teachers, so she had authority behind her. In spite of some lack of spontaneity in all this situation (resulting from the fact that the teacher, probably, had never solved this problem on her own) the students generally understood the idea. A similar problem was given as homework and all diligent students made it.

A similar problem is included in a modern Russian textbook, also for the fourth grade:

Problem 11 An ancient artist drew scenes of hunting on the walls of a cave, including 43 figures of animals and people. There were 17 more figures of animals than people. How many figures of people did the artist draw? [**Geidman.4.1**], p. 11

A similar problem is included in the 5-th grade Singapore textbook:

Problem 12 Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive? [**Sing.5A**], page 23

The problem is followed by visual hints: there are two horizontal bars named Raju and Samy and the data are shown very clearly in this picture. In addition, there is a girl, from whose head a thought appears in a cloud, saying “Give Raju $100 first and divide the remaining money into two units.” Then it is written:

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2 units = $410 - $100
        = $310
1 unit  = $...
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`Samy received $...`

Two pages further a similar problem is given to solve independently:

Problem 13 John is 15 kg heavier than Peter. Their total weight is 127 kg. Find John's weight. [**Sing.5A**], p. 25 $\boxed{\textbf{P : weight}}$

I believe that all this is a piece of good pedagogics. There are many other useful pictures in Singapore textbooks.

Similar problems are solved without algebra in Japan, also in the 4-th grade, according to Stevenson and Stigler's "Learning Gap" [**Gap:L**]. On p. 187 they describe a Japanese teacher explaining the following problem to 4-th graders:

Problem 14 There is a total of 38 children in class. There are 6 more boys than girls. How many boys and how many girls are there? $\boxed{\textbf{P : class}}$

Stevenson and Stigler add that "its solution is generally not taught in US until students take a course in algebra" and write on the next page:

> With this concrete visual representation $< \cdots >$ and careful guidance from the teacher, even fourth-graders were able to understand the problem and its solution.

Well, what does it actually mean, "until students take a course in algebra"? This is what Robert B. Davis wrote in 1989:

> It is probably still true that the most common pattern in the United States is for algebra to be presented as a one-year course, taught in ninth grade, for

> which students will have received relatively little prior preparation in earlier grades. In some schools algebra has become an eighth-grade course, in a few (such as University High School in Urbana, Illinois) it has become a seventh-grade course, and there is a project in Lincoln, Massachusetts, that presents algebra in sixth grade. [**Davis**], p. 267

Although sixteen years have passed since Davis said it, the situation has not changed very much. I have just searched Google for "algebra in eighth grade" and found many references showing that this remains a challenge for American education. For example, the abstract of Frances R. Spielhagen's declaration at the 2004 Annual Meeting of the American Educational Research Association says:

> This study examined early access to algebra in a large urban/suburban school district that provided algebra instruction to some students in eighth grade. Data analysis explored differences in algebra achievement and the background of the two treatment groups, i.e., those who studied algebra in eighth grade versus those who studied it in ninth grade. Data analysis did not support tracking students into two separate treatment groups. Group membership did not guarantee higher achievement but in fact reinforced existing achievement patterns. The results of this study support policies that promote equity by offering algebra to all students in the eighth grade. [**AERA2004**]

Thus, we may hope that several years later, in result of Frances R. Spielhagen and his fellows' efforts, American 8-graders will solve those problems which Japanese, Russian and Singaporean 4-5-graders solve right now.

I believe that a well-explained arithmetical solution of a problem in the vein described in this section, accompanied by some visual representation and followed by one or several similar problems given to solve independently (at home) has a great pedagogical value.

Children of various countries face similar problems around the same age. Since Japan, Russia and Singapore teach this kind of problems around the same age, we may suggest that these countries' curricula fit some general laws of human development: probably, modern children at this age are ready just for problems of this level. However, laws of this sort cannot be absolute, they always depend on social environment.

Now I want to discuss again, why arithmetical word problems are so important. I agree with Kudriavtsev's idea that they promote development of logical thinking, but he does not explain how or why. Shevkin undertook such an explanation in at least one of his books [**Shevkin.T**]. On p. 10 he wrote:

> At this stage of teaching [5-6 grades] arithmetical ways of problem solving have an advantage over the algebraic one, already because the result of every single step in a step-wise solution has quite an evident and concrete interpretation, which does not go beyond the students' experience. It is not by chance that the students master various (even complicated) ways of argumentation, based on imaginary operations with known quantities, faster and better than the way based on usage of an equation, one nad the same for problems with different arithmetical situations.

I think that this is very much true, but want to add the following. When solving problems arithmetically, children build certain representations (like pictures in Japanese or Singaporean textbooks mentioned above). The algebraic method seems to many teachers not to need such representations, so it usually turns into mechanical manipulation of sumbols.

4. Problems by parts

Now we go to another class of problems, also not quite forward: you must find two numbers given their sum and ratio. In Russia such problems are called “problems by parts” because you can solve them without algebra by introducing “parts”. These problems are so common in Russia that a well-known writer, Nosov, described one of them in his book ”Vitya Maleev at school and at home”. The hero, Vitya Maleev, has just finished the third grade. He was weak in mathematics and promised his teacher to train himself in solving problems to catch up. So he tries to solve the following problem from the 3-grade textbook:

Problem 15 A boy and a girl collected 24 nuts. The boy collected two times more nuts than the girl. How many did each collect? **P : nuts**

The author describes Vitya’s process of thinking in great detail. First Vitya divides 24 by two and gets 12. May be, each collected 12 nuts? No, the boy collected more than the girl. Not knowing what to do, Vitya draws a picture of a boy and a girl. To express the fact that the boy collected two times more nuts than the girl, Vitya draws two pockets on the boy’s pants and one pocket on the girl’s apron. Then he looks at his picture and sees *three pockets.* Then an idea “like a lightning” comes to his mind: the nuts should be put into these pockets, so he should divide the number of nuts into the number of pockets! Thus he gets $24 : 3 = 8$. So each pocket contains 8 nuts. This is how many nuts the girl has. The boy has two pockets, so he has 8 times 2, which gives 16. Now Vitya can check his answer: he adds 8 and 16 and gets 24. Now he is sure that his solution is correct. He is very excited. Probably, this is the first time in his life that he solved a problem on his own. He goes to the street to tell somebody about it, but nobody shares his excitement. A neighbor girl says: “This is a third-grade problem. We solved them last year.” This does not diminish Vitya’s joy and he is right: he made a discovery. Let me emphasize that this discovery was possible due to certain qualities of the problem, especially its exact setting. You cannot make a discovery if you are not sure how to tell right from wrong for the same reason why you cannot win in a game without rules. Although Vitya probably would never become a professional mathematician even if he existed in reality, his discovery is genuine in personal terms. Barry Mazur's article [**Mazur**] explains why is it so important.

5. Division... Which Division?

Look at this problem:

Problem 16 An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed? **P : soldiers**

This problem was included in one of the NAEP (National Assessment of Educational Progress) secondary mathematics exams. Alan Schoenfeld [**Schoenfeld.F**] wrote about it:

> Seventy percent of the students who took the exam set up the correct long division and performed it correctly. However, the following are the answers those students gave to the question of "how many buses are needed?": 29% said... "31 remainder 12"; 18% said... "31"; 23% said... "32", which is correct. (30% did not do the computation correctly).

Such problems have been included in Russian elementary school problem books. This is problem 702 on p. 152 of the Russian textbook for the 4-th grade. Moscow, 1992 [**Moro.4old**].

Problem 17 Each box can contain 20 kg of carrot. How many boxes are necessary to transport 675 kg of carrot? **P : carrot.1**

I have no doubt that Russian children, when meeting with such problems first time in their life, made the same ridiculous mistakes as mentioned by Schoenfeld. But pay attention to the difference of ages: Russian children encounter such problems already in the 4-th grade, when they are ten years old. At that innocent age it is not fatal to make ridiculous mistakes.

In this connection let us consider also the following problem:

Problem 18 A company packs carrots into boxes. Each box must contain 20 kg of carrots. The company has 675 kg of carrots. How many boxes can it fill with carrots? **P** : **carrot.2**

Although this problem is very similar to the previous one, its answer is 33 boxes, not 34, although all the given numbers are the same and the operation also seems to be the same: division. This example shows again that word problems are not only mathematics. They are also something else and this **heterogenity** of word problems adds a special kind of difficulty to them and makes them especially useful for general development.

6. Schemes

Now we are going to present a partial answer to the question asked in the first section: what besides numbers, variables and equations results from "translation" of word problems? Remember pictures in the Japanese and Singaporean textbooks which helped children to imagine the situations described in the problems. Remember also the picture drawn by my teacher. Remember the picture with three pockets drawn by Vitya Maleev. All of them are instances of a general tendency to use visual representations when solving word problems. I believe that such visual or spatial representations are essential for productive work in mathematics. Students called “uncapable for mathematics” are those who for some reason fail to develop such representations and often don’t even know that it is desirable or possible. They try to solve problems doing something with the very words without translating them into something else.

All this may seem natural or even self-evident, but inclusion of useful schemes in a textbook takes professional maturity and good will. Too many American textbooks are filled with irrelevant pictures. which only distract children’s attention and don’t help them to think. This is what Martin Gardner wrote about a textbook known as “Rainforest algebra”:

> Many pictures have only a slim relation to the text. Magritte’s painting of a green apple floating in front of a man’s face accompanies some problems about apples. Van Gogh’s self-portrait is alongside a problem about the heights and widths of canvases. A picture of the Beatles accompanies a problem about taxes only because of the Beatles’ song ”Taxman.” My favorite irrelevant picture shows Maya Angelou talking to President Clinton. [**Gardner**]

I believe that visual or spatial representations are essential for success even when we deal with formal objects, such as formulas and equations. An example. It was probably our last year in Russia. My daughter’s school group started algebra. The homework was to solve an equation $A - X = B$, where A and B were some numbers, which I forgot. My daughter had to find X, but she had no idea how to do it. I put my hands vertically on the border of the table and said: “Look, this is A”. Then I moved one hand half-way towards the other and said: “Now X is cut from it and what remains is B. How to find X?” My daughter said: “Ah, I must subtract.” All this took place in the kitchen of our Moscow apartment. at the table where we cut bread every day, and when I cut X away from A, it reified those abstract notions for my daughter. After that she easily solved equations assigned in school.

7. Problems which can be solved using an ad hoc unit

A still more sophisticated kind of word problems are those, which need an ad hoc unit to solve. Let us remember the problem 1 $\boxed{P: books}$ for the 4-th grade:

A library needs to bind 4500 books. One shop can bind these books in 30 days, another shop can do it in 45 days. How many days are needed to do the job if both shops work as the same time?

However, this problem can be solved without knowing the number of books. We don't even need to know what kind of work needs to be done. It is sufficient to know that the first shop can do $1/30$ of the work in one day and the other one can do $1/45$ of the work in one day. Therefore, working together, they can do this part of the work in one day:

$$\frac{1}{30}+\frac{1}{45}=\frac{3}{90}+\frac{2}{90}=\frac{5}{90}=\frac{1}{18}.$$

Therefore they need 18 days to do the work.

Problems which needed an ad hoc unit were considered rather difficult for the regular school studies. However, for a mathematical circle or an olympiad they were appropriate.

Now I found a similar problem in a textbook for the 4-th grade. It is much easier, so it can serve as an introduction into "work problems":

Problem 19 Deniska can eat a jar of jam in 6 minutes. Mishka can eat a similar jar of jam two times faster. In how much time will they eat a jar of jam together? [**Geidman.4.1**], p. 34.

Deniska and Mishka are colloquial versions of common Russian names, which fit into the jocular style of this problem.

The tradition of using work problems is old and has respectable names behind it. For example, Polya [**Polya**], p. 47 cites the following problem from Newton's textbook:

Problem 20 Three Workmen can do a Piece of Work in certain Times, viz. A once in 3 Weeks, B thrice in 8 Weeks, and C five Times in 12 Weeks. It is desired to know in what Time they can finish it jointly. $\boxed{\textbf{P : Newton}}$

The solution which both Newton and Polya had in mind is based on the same assumption as before: that each workman has a constant rate. We can take the 'Piece of Work' mentioned in the problem as a unit of work and call it 'job'. Then A's rate equals 1/3 job/week, B's rate equals $3/8$ job/week, and C's rate equals $5/12$ job/week. When they work together, their rates add, and the total rate is

$$\frac{1}{3}+\frac{3}{8}+\frac{5}{12}=\frac{9}{8}.$$

Then the time they need is 1 job divided by $9/8$ job/week, that is $8/9$ of a week.

It is interesting to think about schemes used by good solvers of such problems. I suggest that such schemes need some "elasticity" to represent the fact that certain quantities cannot be determined.

8. From arithmetics to algebra, geometry and physics

Let me present a few more problems from Russian books for grades 6-8 (12-14 years old). The following problem is especially usefil first to solve by parts, without algebra, and then to solve again, using algebra.

Problem 21 An ancient problem. A flying goose met a flock of geese in the air and said: "Hello, hundred geese!" The leader of the flock answered to him: "There is not a hundred of us. If there were as many of us as there are and as many more and half as many more and quarter as many more and you, goose, also flied with us, then there would be hundred of us." How many geese were there in the flock? [**Larichev**], P. 37.

P : geese

The following problem was included in a Russian school textbook. It shows the power of the physical idea of relative movement.

Problem 22 (HISTORICAL PROBLEM.) A swimmer was swimming upstream Neva River. Near the Republican Bridge he lost an empty flask. After swimming 20 min more upstream he noticed his loss and swam back to find the flask; he reached it near the Leughtenant Schmidt Bridge. Find the velocity of current at Neva River if the distance between these two bridges is 2 km. [**Larichev**], P. 208. **P : swimmer**

Let us place ourselves in the coordinate system moving with the stream. In this system the flask does not move when it is lost, while the swimmer swims first away from it, then towards it. His proper velocity is assumed constant, so he spends one and the same time swimming in both directions. But one of them took 20 minutes, so the other also takes 20 minutes, so the total time when the flask was lost is 40 minutes. Now we return to the coordinate system where we were before, and notice that the flask took 40 minutes to move from one bridge to the other, that is 2 km. So the velocity of the current is 2 km divided by 2/3 hour, which makes 3 km/h.

This is a (slightly changed) word problem with geometric content, included in one of Perelman's books:

Problem 23 A man sold firewood. To make standard portions, he always used one and the same rope, surrounded a pack of logs with it and brought it into houses on his back. One woman asked him to bring a double portion of firewood. The man proceeded as usual, only took a rope one and a half times longer than usual. The woman complained: "Since

I payed you a double fee, you should use a double length rope." The man replied: "You are mistaken, mam. In fact, I brought you even a little more firewood than you requested." Who is right? [**Perelman.P**], P. 27 **P : firewood**

To solve this problem, we, as usual, have to make simplifying assumptions. We assume that the firewood surrounded by a rope is a cylinder, whose height is the length of the logs and base's circumference equals the lenght of the rope. Since the height of the cylinder is constant, the volume of the cylinder is proportional to the area of the base, which is proportional to the square of the radius or, which is the same, to the square of the circumference. So, if the length of the rope is multiplied by 3/2, the amount of the firewood is multiplied by a square of this amount, which is 9/4, which is really a little more than 2. The man was right.

Problem 24 A PROBLEM FROM ARNOLD'S INTERVIEW [**Arnold**]. Two old women started at sunrise and each walked at a constant velocity. One went from A to B and the other from B to A. They met at noon and, continuing with no stop, arrived respectively at B at 4 p.m. and at A at 9 p.m. At what time was the sunrise on that day? **P : Arnold**

To solve this problem, one may draw a diagram with distance from A and time as coordinates and use similarity of triangles. Vladimir Arnold, a famous mathematician, emphasizes that solving this problem independently (in 1949 when he was 12 years old) was his first real mathematical experience, uses the words *revelation* and *feeling of discovery* to describe this experience and says that in Russia his experience was not unusual.

Indeed, usage of arithmetical word problems is traditional in Russia. The following problem is more than a hundred years old.

Problem 25 A team of mowers had to mow two meadows, one twice as large as the other. The team spent half-a-day mowing the bigger meadow. After that the team split. One half of it remained at the big meadow and finished it by the evening. The other half worked on the smaller meadow, but did not finish it at that day. The remaining part was mowed by one mower in one day. How many mowers were there? [**Perelman.A**], p. 39

P : Tolstoi

Perelman calls it *Tolstoi's problem* because Leo Tolstoi, the famous Russian writer, who was very interested in public education, liked it.

More about arithmetics in Russia in 1930s-1950s. The 20-th edition of the book [**Berez**] was published in 1953. It was entitled "Collection of problems and exercises in arithmetics" and indeed most of its problems are forward, that is can be solved by arithmetical means. Such problems may contain several steps. This is an example:

Problem 26 When milled, wheat loses 1/10 part of its weight. How much bread can be obtained from $1\frac{1}{2}$ ton of wheat if when bread is baked, the surplus weights 2/5 of the flour used? [**Berez**], p. 114. **P : B : bread**

However, [**Berez**] contains some problems, which are not so straightforward. For example, Leo Tostoi's problem is included there on p. 248.

The book contains problems, which are much more easy to solve using algebra than without it. However, all these problems were supposed to be solved arithmetically. This is a problem from the very end of the book:

Problem 27 The mother is $2\frac{1}{2}$ times older than the daughter. Six years ago the mother was 4 times older than the daughter. How old are the mother and the daughter? [**Berez**], p. 270. **P : Ber : age**

This problem is followed by an indication:

> Since the mother is $2\frac{1}{2}$ times older than the daughter, the difference of their ages contain the daughter's age $1\frac{1}{2}$ times $\left(2\frac{1}{2}-1\right)$. Six years ago the same difference contained the daughter's age 3 times $(4-1)$. So in 6 years the daughter's age increased 2 times $\left(3 \div 1\frac{1}{2}\right)$.

This tampering with numbers probably was too much. In Larichev's books used in the next decades it was reduced and algebra was taught instead of it. However, while arithmetical solutions are elegant enough, they should be mastered by every child. I remember that I liked to find arithmetical solutions of word problems. Using algebra seemed bad sport to me.

In the subsequent years the percentage of children attending school increased and correspondingly the level of problems somewhat decreased. Larichev's textbook [**Larichev**] for 6-8-th grades (12-14 years old) used in 1950s-1960s is somewhat more easy than was [**Berez**] for 5-6 grades. However, Larichev's book was substantial and useful. Here are a few problems from it :

Problem 28 The rectangular lid of a box has length 30 cm and width 20 cm. A rectangular hole with the area 200 square cm must be made in this lid so that its sides were at equal distances from the sides of the lid. Which should be the distances of sides of the hole from the sides of the lid? [**Larichev**], p. 259.

Most problems in this section were considered somewhat advanced in Russia, slightly beyond the regular school course, but not too far from it. Larichev's textbook [**Larichev**] for 6-8 grades, which was used when I was at school, consisted mainly of more standard problems, but a sample of more advanced problems was available in Larichev's book, so that a curious student living in a remote region could find interesting problems to solve. At that time Larichev's textbook seemed an embodyment of mediocrity to me. Now, after several years of teaching American undergraduates, I am astonished by the high level and quality of Larichev's work. Even problems that look standard in the context of his book, would be considered too hard by most of my American undergraduate students.

I don't mean that the situation with mathematical education in Russia always was perfect. Just one example from 1930s. My father was born in 1921. When he was in high school, all the students were divided into teams and it was assumed that students in a team study together, because according to the principles of communism people should work together. Since students studied together, it was not necessary to check everyone's knowledge in every subject: in was sufficient to test one representative of every team. In practice members of a team divided all school subjects between themselves, so that every student studied only what he liked. For example, my father studied only literature and history, then easily passed exams and his note was attributed to all the members of his team. Another member studied only exact sciences etc. This method proved very efficient in elimination of failures: since it was introduced, all the students got only good grades. However, several years later this "communist" method was abolished.

9. Russian standards

Under Communist rule the Ministry of Education explicitly specified, what should be taught every school year. (Remember that girl who easily identified the problem 15 $\boxed{P : nuts}$ as 3-d grade problem.) Since the collapse of the Communist rule, mathematical education in Russia did not become much better or much worse. The National Standards [**Rus.Fed**] now include only goals to be achieved, but do not specify when and in which order the topics must be taught. I am not sure that it is better. Also it is very important that there are excellent classics including books by Perelman and there are new authors including Shevkin. After all, many students prepare for college, where they will have to pass entrance examinations consisting largely of solving problems.

The modern Russian Standards document for high school [**Rus.Fed**], p. 81-91, consists of several parts. It is ridiculous that the first part "Expressions including numbers and letters" does not mention solving problems at all. However, the tradition of solving word problems through all school grades is so strong in Russia that this omission seems not to harm the practice of teaching.

only first?

However, all this may change as years pass on. In my opinion, it is essential for school mathematics standards to contain samples of problems which students are expected to be able to solve by the end of every year. I shared this idea with Vitaly Arnold, a prominent Russian educator and he agreed with me. In fact, he and his colleagues had made such a proposal to the Ministry of Education and got a refusal for the following reason. The Ministry wanted stndards for different subjects look uniform. If it were possible to include problems into standards for all subjects, the Ministry might agree. But solving problems plays such a promonent role only in Mathematics, well also Physics, well perhaps also Chemistry. Other subjects are taught in other ways. So including lists of problems into standards would violate uniformity, which is unacceptable for the Russan Ministry of Education.

10. Word problems at mathematical olympiads and circles

throughout my undergraduate years I participated in organization of mathematical olympiads. The school teachers all the time reproached us for using too difficult problems. However, the gap between school and olympiads and circles was not really so deep. This is an example. This problem was used at Moscow Mathematical Olympiad in 1963 in the first tour of 9 grade.

Problem 29 Given a rectangle whose sides relate as 9:16. Is it possible to inscribe into it another rectangle, whose sides relate as 4:7, in such a way that every side of the first rectangle will contain one vertex of the second rectangle? **P : rectangle**

The answer is negative, which can be proved by contradiction. Let us assume that it is done. Due to similarity of triangles, we can denote the parts of sides of the big rectangle by $4x, 7y, 4y, 7x$. Then we observe that $4x + 7y : 4y + 7x = 9 : 16$, which is impossible, since x and y must be positive. Only about half of 9-graders, who attended the first tour, solved this problem. As usual, after every tour solutions of all problems were explained in a big auditorium. In this case it was full. Addressing to about 500 young snobs, who thought that school textbooks were too trivial for them, I shocked them by quoting a similar problem from the school problem book. Only the numbers were different, so the school problem had a positive answer.

The role of word problems as stepping stones towards theory is even more visible at olympiads and mathematical circles. The following problem appeared first in a book by Perelman, then at a Moscow mathematical olympiad in 1940 and a few years later was included into a school problem book for 5-6 grades:

Problem 30 A boat, going downstream, made a distance between two ports in 6 hours and returned in 8 hours. How much time is needed for a raft to make this distance downstream? [**Berez**], p. 246 **P : raft**

We can solve this problem using an *ad hoc* measure of distance: one *trip*, which equals the distance went by the boat in one direction. This provides us with a unit of velocity, one *trip per hour* or *trip/hour*. Due to this, we can denote by X and Y the boat's proper velocity (that is, velocity in still water) and the current's velocity in trips per hour. The boat's actual velocity is X+Y downstream and X-Y upstream. So we can write the equations $6(X+Y)=1$ and $8(X-Y)=1$, whence we find that $Y=(1/6-1/8)/2=1/48$ trip/hour. This is speed of the current, but the raft's speed is the same. Then the time needed by the raft to make one trip equals one trip divided by 1/48 trip/hour, that is 48 hours.

There are many collections of interesting problems in Russia, not all of which have been translated into English. This is a problem from a translated book:

Problem 31 Katya and her friends stand in a circle. It turns out that both neighbors of each child are of the same gender. If there are five boys in the circle, how many girls are there? [**Circles**], p. 6 **P** : **Katya**

Notice that Katya is a girl and it is essential!

This is a problem from another book which has been translated:

Problem 32 Fourth-grade pupil Kolya Sinichkin wants to move a knight from the lower left corner of the chessboard (a1) to the upper right corner (h8), visiting every square en route once. Can he? [**Kordem**], p. 48 **P** : **knight**

This is a problem from a book which has never been translated:

Problem 33 Three friends played chess so that every two of them played one and the same number of parties. After that they argued who is the winner. The first one said: I won more parties than each of you. The second one said: I lost less parties than each of you. The third player said nothing, but when the points were counted, they found that he had gained more points than each of the others. Is it possible? (A victory brings a point, a draw brings half a point, a loss brings nothing.) [**VGRT**], p. 106 **P** : **chess**

Word problems used at more advanced studies and stages of olympiads serve the same function with respect to more advanced ideas: they implicitly introduce children into substantial mathematics, for example number theory, graph theory or combinatorics without boring introductions or heavy terminology. This shows how fruitful is the practice of mathematical circles, where interesting and substantial mathematics is taught in such a manner that children in a relatively short time, without any pomp, become proficient and creative. For example, if you want to introduce children into graph theory, you don't need to start with cumbersome terminology and definitions. Instead you can give them a problem:

Problem 34 $2n$ knights came to King Arthur's court, each having not more than $n-1$ enemies among the others. Prove that Merlin (Arthur's advisor) can place the knights at a round table in such a manner that nobody will sit beside his enemy. [**GT**], p. 89 $\boxed{\textbf{P : Arthur}}$

This problem was proposed for the 27th Moscow Mathematical Olympiad, but it was unusable in its original form: *A graph has $2n$ vertices, each vertex being incident with at least n edges. Prove that this graph has a hamiltonian cycle.* I proposed to represent the hamiltonian cycle by the legendary round table, and in this form the problem was accepted. After that it was discussed at mathematical circles, where knights were represented by circles and friendship relations by lines connecting them. Thus discussion of a "jocular" problem smoothly turned into a study of graph theory, which was non-trivial from the very beginning.

When American educators speak about mathematical education in Russia, they usually remember some special projects, such as mathematical schools [**Shen**] and circles [**Circles**]. (See also my review [**Toom.Circ**].) Indeed, all of them were and are useful, but I want to say something else. When I taught mathematical circles where we solved non-standard problems, I took for granted my students' basic knowledge and skills and ability to solve standard problems. Without all this we, university students, unexperienced in teaching, would not be able to teach advanced topics.

When I came to America, I taught several classes of problem solving and started to appreciate much more the basic education, because my new students dramatically lacked it. They understood advanced ideas but floundered in algebraic transformations, which turned solution of interesting problems into painful struggle with basics. For example, when we studied the method of mathematical induction, my American students understood the idea (which is not trivial), but got into trouble when they needed to substitute $n+1$ instead of n into a formula. This showed to me that there was something wrong with the most basic level of mathematical education in America. In this I am not alone as you will se below.

In Russia I had another useful experience: collaboration in the School by Correspondence, most students of which lived in villages and small towns and were culturally deprived by comparison with children of metropolitan areas. In addition, we could not meet with them face to face. We sent them brochures which included problems, the students solved them as they could and sent their solutions to us. We checked and commented their solutions and sent our comments to them, after which the students could try to solve the same problems again. For example, the following problem was used:

Problem 35 Is there a triangle, whose area is greater than 100 square meters, but every height is less than one centimeter? $\boxed{\textbf{P : triangle}}$

This problem has at least two merits: First, it moves students to expand their supply of particular triangles with various properties. Second, it helps to distinguish between a vague idea and an exact description. Some students replied: “Yes, there is such a triangle. It should have a very large base and a height equal to only a few millimeters.”

Such a bizarre amalgama of insight and confusion is typical whenever students solve problems on the border of their possibilities and a major problem in mathematical education is how to react to it. On one hand, the student who hit at this idea should be encouraged, because it is a valuable insight, but on the other hand it would be a very bad service to that student not to make him aware how much this insight is short of a mathematically correct solution. On request of the School by Correspondence I wrote instructions for its teachers. In this case I recommended to answer in the following vein:

> What you wrote is a valuable *draft*, which may lead to a solution, but it is not yet a solution. Now, based on your draft, describe some concrete triangle exactly and show that its area and heights satisfy the required inequalities. Only having done this you will solve the problem!

Problems of this sort do not yet involve explicit proofs, but they develop mental discipline which is necessary to deal with proofs. It seems to me that such preparations are rare in USA.

From olympiads, circles and special schools we naturally go to research. Roland Lvovich Dobrushin and Yevgeniy Borisovich Dynkin are prominent Russian specialists in probability. Dobrushin first encountered with probability at a seminar led by Dynkin. Then Dynkin emigrated to USA and worked at Cornell University. He liked to record interviews with his visitors. This is what Dobrushin told Dynkin in his interview:

> I remember a problem, due to which I understood what is probability theory. This problem was set for us, who did not yet know the word "probability". Here it is: there are N vessels, and from each some fraction of water goes to another and it must be proved that the quantity of water in a given vessel tends to a limit. Honestly speaking, I think that for me the probability till

> now remains water or liquid, which goes from one vessel to another. This is how I imagine a random process. **[Dobr]**, p. 9

I think that what Dobrushin says here is very important for all stages of doing mathematics from elementary school to research. We, mathematicians, publish articles only when we have complete proofs. But how do we find proofs? Certainly not by listing all logically valid arguments! We have some vision of the situation. It is very important to help children to develop such visions. In this sense Vitya Maleev's discovery of three pockets, Vladimir Arnold's solution of the walkers' problem and Dobrushin's vision of probability as liquid belong to one and the same category.

11. The Traditional Way of Posing Word Problems

Which features of traditional word problems make them so useful to develop mathematical vision?

The most indispensable feature of the traditional word problems (and mathematics problems in general) is careful editing. This means that all ambiguities must be excluded, so that to make it clear what is given and what is asked. Also it must be clear that school word problems as well as all instances of application of mathematics do not deal with real objects themselves. They deal with *models* of these objects, which are always simplified because we cannot divide our attention between all the many features of reality.

This requirement of clarity and economy of attention is not restricted to mathematics. It belongs to the general tradition of good thinking and good presentation. When you read directions for travellers, you expect them to be clear and precise and not to encumber your attention with irrelevant details. Geogrpahical maps cannot and should not be equal to those landscapes which they represent. The same refers to an instruction how to use an appliance. In all these cases you expect clarity and concentration on the relevant features. An unclear law may is dangerous and a law involving irrelevant detail is inefficient. I would even say that clarity and condensation are permanent requirements of civilization.

This implies the following features of traditional word problems.

1) All traditional word problems describe an imaginary situation, which is intrinsically consistent (that is, does not contradict itself), but does not need to be realistic. They provide certain data about this situation and ask a question (or questions), the answer to which is determined by the data.

2) It is always assumed that the answer should be based only on the provided data plus certain usual assumptions. These assumptions must be easy to formulate (like constant rates of functioning) and clear to teachers and students. If more than one numerical answer fits these data and assumptions, the answer should be presented as a set, which includes all those values of the quantity in question, which do not contradict the data and the usual assumptions. If none number fits the data and the usual assumptions, then the students should say "there is no answer" or present the empty set as the answer.

3) To save the reader's attention, these problems usully avoid irrelevant data.

4) These problems avoid cumbersome numbers to help the students to concentrate on the conceptual difficulties. Thus the numbers and calculations usually present less difficulty than decisions what to do and in which order.

All these qualities are typical of those countries, which get high scores at international comparisons. For example, I browsed several school textbooks from Singapore [**Sing.3A, Sing.4A, Sing.5A, Sing.6A**] and found plenty of word problems, whose difficulty continuously grew from one grade to another, but none exception from these rules.

All of these problems have a unique numerical answer, which can be obtained applying certain arithmetical operations to the data. No extra data are needed.

None of these problems has an extra datum. In other words, you need all the data to solve a problem.

All the data are not too complicated and all the calculations are possible within students' competence.

I believe that all these features are essential for efficient schooling.

Part II. Word problems in America

12. Ignorance of American Teachers

In my opinion, the most important factor contributing to the ailments of American mathematical education is dire ignorance of many American teachers of mathematics. I don't mean that competence of teachers in other countries is perfect. When I studied at Moscow University, we, students involved in teaching mathematical circles and organization of mathematical olympiads, giggled a lot about school teachers' rigidity and inability to cope with a slightly unusual situation. Olympiad-style problems, which were easy for us, were difficult for most school teachers. But we could not imagine a certified teacher ignorant in basics of school program. In USA such ignorance is pretty common, perhaps because nobody knows what is school program.

I would even say that the main efforts of some leaders of American mathematical education in the last decades was not to improve the quality of teachers, but to distract the public attention from teachers' ignorance. Some documents issued in this period are vague and ambiguous; one can read admissions of poor teachers' preparation between their lines, but not in the text. For example, on p. 126-127 the 1989 "Standards" recommend to increase attention to "deductive arguments expressed orally and in sentence or paragraph form" and correspondingly to decrease attention to "two-column proofs" as if form of presentation of arguments made any difference. It would be more honest to say that teachers are dreadfully unprepared to teach proofs in any form, would it be oral, paragraph or two-column.

However, now we have some open declarations about unpreparedness of teachers, although none of them appeared in an educational journal. One of them was Liping Ma's famous book [**Ma.book**]. I don't comment it because it is alredy well-known. Another is the article published lately by Patricia Clark Kenschaft in the Notices of AMS [**Kensch**] (not in an educational journal). The article is full of eye-opening examples. This is one of them (p. 209): in 1986 Kenschaft visited a K-6 elementary school and discovered that *not a single teacher* knew how to find the area of a rectangle:

> "What is the area of a rectangle that is x high and y wide?" I asked. $< \cdots >$ "x plus y?" said two in the front simultaneously. $< \cdots >$ Then all fifty of them shouted together, "x plus y."

Although Kenschaft's main concern is education of Black children, she observes that mathematical knowledge of teachers of schools with mostly white students is poor also (p. 210):

> My first time in a fifth grade in one of New Jersey's most affluent districts (white, of course), I asked where one-third was on the number line. After a moment of quiet, the teacher called out, "Near three, isn't it?" The children, however, soon figured out the correct answer; they came from homes where such things were discussed. Flitting back and forth from the richest to the

poorest districts in the state convinced me that the mathematical knowledge of the teachers was pathetic in both. It appears that the higher scores in the affluent districts are not due to superior teaching in school but to the supplementary informal "home schooling" of children.

Two years ago The Education Trust issued a report on how States of the Union fulfill the NCLB (No Child Left Behind) program launched by President George W. Bush [**Telling**]. The focus of this document is not yet how they improve the situation, but only how they report about that sad reality that many subjects including mathematics are taught by incompetent teachers, especially in the areas populated by poor people. The document shows that some states simply report nothing, thereby ignoring explicit requests from the Federal Department of Education. Some states answer questions different from those which were asked and some states report data which evidently cannot be true.

13. Arithmetical word problems in USA, more exactly their absence

In USA the situation with word problems, especially arithmetical word problems is much worse than in Russia. The very word "arithmetics" is avoided by some American educators because they are afraid of looking provincial. Of course, this fear is exactly what makes them provincial. Lately Liping Ma made a public speech [**Ma.talk**], even the title of which was polemical:

Arithmetics in American Mathematics Education:
An Abandoned Arena?

In it Ma presents several simple word problems taken from textbooks of Russia and Singapore and other sources, explains some of their merits and finally exclaims:

> How was the arena of arithmetic, as taught in other countries, abandoned in the U.S., and why? I believe that there must be some positive reasons that made it happen. However, serious reflections on this issue need to be conducted.

That Liping Ma is right, I can confirm by a personal observation. My daughter was always interested in art more than science, but she never was afraid of mathematics and easily coped with the school program in Russia. This program included many several-step word problems, which my daughter (like most Russian children) did not find especially difficult. She started to attend an American middle school when she was twelve years old. There were three groups in math, – slow, middle and fast, and she was placed in the middle. She immediately noticed that all the problems they solved were one-step. Even worse: the teacher explained how to do an arithmetical operation and then they solved problems on that operation. My daughter asked to move her to the fast group, but was not because of her poor English. Several months later she got to the fast group, but found there also only one-step problems. The only difference between the groups was the following: in the middle they solved one-step problems in the usual sense, in the slow group the children were given problems with the answers and had to check them and in the fast group they were given problems incompletely and had to ask a question and answer it. I conclude from this that the school simply had neither curriculum, nor textbook, nor tradition to follow for a really advanced group. And most probably no teacher too.

Was that school an exception? I don't think so. There are many indications that this situation is typical: most mathematics studied in American public middle schools is simply repetition of what was studied in elementary school and what is studied in elementary school is not a great thing also. Katherine K. Merseth in her essay [**Merseth**] quotes a teacher saying: *"If Johnny doesn't get multiplication in third grade,* *he'll have another chance in fourth, fifth, sixth, seventh, and eighth grades".*

It is typical of American educators to classify word problems into four categories: addition problems, subtraction problems, multiplication problems, and division problems. Of course, only one-step problems fit into this classification, so all the others have no place to exist.

There is only one journal in USA, explicitly devoted to research in mathematical education. It is small and thin and never mentioned coin, digit, work, age or any other useful type of word problems on my memory, but may devote its scarce place to a sample of one-step problems like this:

Problem 36 To raise money for equipment, the Pioneer Hockey Club decides to have a car wash. 29 members volunteer to wash cars over the weekend. On Saturday the weather is pretty cold, but people still bring their cars to be washed. On Sunday the club washes 67 cars. Over the entire weekend the hockey club washed 91 cars. How many cars did they wash on Saturday ? [**JRME**], p. 479 **P : carwash**

14. Attitudes towards word problems in USA

In America word problems (also known as story problems or verbal problems) are not taken as easily as in Russia, on the contrary they arouse mixed, even contradictory feelings. On one hand, word problems are considered especially difficult, even frustrating. Generally, frustration caused by mathematics is one of the most noticeable concerns of American educators, which is understandable because schools of USA have no established curriculum, so very often there is no continuity in the courses a student takes: some courses repeat each other, some are separated by gaps; in the former case boredom and in the latter case frustration is inevitable. Word problems seem to create the greatest frustration. I think, it is because their difficulty is not so evident (a good deal of it is in syntax and semantics of the natural language rather than in a well-established, recognizable mathematical theory), so in their case American administrators are especially careless about continuity. There is a cartoon in the Far Side series called "Hell's Library" showing a library full of story problem books. Mildred Johnson, an experienced teacher, starts her book about word problems (actually very easy ones against Russian standards) as follows: "There is no area in algebra which causes students as much difficulty as word problems." [**Johnson**] Pay attention that even Professor Johnson, in spite of her competence, thinks that word problems are part of algebra. She seems not to be aware of arithmetical word problems.

There are many books on word problems at American book market now. Typically they warn you that word problems are very difficult, but assure you that they will become easy for you if you buy that particular book, which is promoted. At `amazon.com` Johnson's book is presented as part of "How to Solve Word Problems Series" and is is said that customers who bought this book also bought... then follows a long list of books that have the magical power to make word problems easy for you.

I came to USA in 1990 and almost at once started to participate in e-mail discussion lists devoted to mathematical education. Some of these lists were completely recorded and all messages sent to them are available at the internet, so I may quote them. One of these lists was called **math-teach**. To find messages sent to this list sorted in the chronological order, go to

`http://forum.swarthmore.edu/epigone/math-teach/all`.

Also you can search this list for words and phrases. You will find several relevant messages searching it for "word problem frustration". Under the subject line "speed of a current river", you can find a discussion of a problem similar to problem 22 $\boxed{P: \mathit{swimmer}}$. In Russia this problem was always considered elegant and intriguing, but in America it was otherwise. One participant observed that this problem caused too much frustration and proposed to solve some other problems instead of it. I expressed my astonishment and then Don Coleman, an experienced teacher, replied: *Andre,* $< \cdots >$ *it is perfectly clear that such problems cause frustration. It is not a matter of explaining why it is. It just happens.*

On August 20, 1996 I sent to this list several messages previously collected by me, all written by competent teachers. These are a few quotes from them:

> ERIC L. GREEN: Somebody asked why word problems were so rare in math textbooks. The reason should be obvious – they scare elementary school teachers to death. Why do you think we had the discussion on 'keywords'? That's just one device elementary school teachers use to 'insure success', i.e. eliminate the need for kids to struggle and think. $< \cdots >$ I've noticed that many high school teachers avoid word problems for the same reason – to avoid frustration. In this case, kids have been trained for years to view math as a set of algorithms for solving particular problems.

> LYNN NORDSTROM: As an elementary teacher who works with many other elementary and middle school teachers, I agree with Eric when he says word problems 'scare teachers to death' and that provide a lot of frustration for all students. In my discussions with teachers, I find that many of them feel this way because they feel unprepared to teach mathematics.

> MARK PRINISKI: Oooo... Word Problems! Why aren't there more of them in the text? Here's a story from the past... The first year I taught Algebra (17 years ago) I was approached by my department head. He told me that the rest of the math department just skipped the word problems because they were too difficult for the students. (Some of my students would like me to take his advice :))

In addition, on May 10, 1994 Lynn Nordstrom sent to the list this charming joke:

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      ``Student's Misguide to Problem Solving'':

Rule 1: If at all possible, avoid reading the problem.
        Reading the problem only consumes time and
        causes confusion.

Rule 2: Extract the numbers from the problem in the order they
        appear. Be on the watch for numbers written in words.

Rule 3: If rule 2 yields three or more numbers, the best bet is
        adding them together.

Rule 4: If there are only 2 numbers which are approximately the
        same size, then subtraction should give the best results.

Rule 5: If there are only two numbers and one is much smaller
        than the other, then divide if it goes evenly --
        otherwise multiply.

Rule 6: If the problem seems like it calls for a formula, pick a
        formula that has enough letters to use all the numbers
        given in the problem.

Rule 7: If the rules 1-6 don't seem to work, make one last
        desperate attempt.  Take the set of numbers found by
        rule 2 and perform about two pages of random operations
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using these numbers.  You should circle about five or
six answers on each page just in case one of them
happens to be the answer.  You might get some partial
credit for trying hard.
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Although it is a joke, regretfully it is close to reality. This article contains several confirmations of this.

It may look like a contradiction, but at the same time word problems have a reputation of being boring and trivial. The following statements were also made at e-mail lists.

> MARK SAUL, A FAMOUS TEACHER, WROTE: "In New York State we have Regents exams, which kids must take at the end of certain courses. For years, these problems were staples on the exams. To get good grades, and so that the teacher could look good, there were review books published, which gave specific methods for solving these problems. For example, in an upstream-downstream problem, you have a 2x3 array of boxes (distance, rate, time across the top; upstream and downstream along the side). All the students had to do was memorize the labels for the boxes $< \cdots >$"

> RALPH A. RAIMI, ONE OF THE AUTHORS OF [**Fordham.RB**], REMEMBERS: In 1937 I was in the 9th grade and learned word problems, "story problems" as they were called. My teacher hadn't the foggiest notion that mathematics was written or thought about in English, and believed that story problems were like French regular verbs. They came in four types, each with its own endings $< \cdots >$

> JUDITH ROITMAN, ONE OF THE AUTHORS OF [**PSSM**], REMEMBERED: The high school algebra course I took (honors, too) was nothing but imposed

> charts and imposed algorithms. It was boring boring boring and if anyone had told me I'd grow up to be a mathematician I'd have laughed at them."

> JOSEPH G. ROSENSTEIN, ONE OF THE AUTHORS OF NEW JERSEY STATE STANDARDS, PUT IT THIS WAY: Although we mathematicians respond positively to the category of 'word problems', many teachers and students react quite differently to that phrase. In the world of education 'word problems' have come to mean repetitive, stylized, and irrelevant problems which are to be solved using a mechanical and often memorized method. Consequently, if our goal is for teachers and students to solve 'word problems', we need to use a different phrase or find ways to bridge the gap between the different ways that the phrase is being understood.

I think that the reason of this seeming contradiction is that word problems have an enormous potential of variability, and exactly for this reason they are often reduced to a few types which can be solved mechanically. This is done in America, but not in Russia, where the *quantity* of word problems to solve is simply too large to reduce them to types, so that nobody ever tried to do this.

Many American students are so unprepared to solve word problems that even a small unexpected twist makes these problems unsolvable for them. My undergraduate students in America insisted that every problem on the test should be preceded by solving the same problem in class, only with different numbers (see section 15 $\boxed{How}$ based on [**Toom.How**]). Even change of a sign made a problem different for them, for example when a train moved in the opposite direction or a pipe drained water from a pool instead of bringing it there.

Thus, to keep problems solvable, educators had to reduce them to a few types, everyone of which could be solved using a standard method and this is what the quoted authors observed. Their observations helped me to understand American educational documents, which are written in quite a different manner than Russian ones. Russian educational standards are mostly lists of topics to be studied. In America such lists are scornfully called “laundry lists”. American educational documents are not so clear, at least those which are issued by NCTM – National Council of Teachers of Mathematics, a very powerful organization.

I must explain, what are **key words**. I have never heard this phrase in Russia. These words are used by American teachers and students to decide, which operation to apply. For example, if you see the word “more”, probably you should add; if you see “less”, subtract.

From Kenschaft’s observations [**Kensch**] we know that even some schools of education teach to use them. Guides how to use key words abound in USA, including internet. At the address
`http://purplemath.com/modules/translat.htm`
I found the following table:

Addition	increased by more than combined, together total of sum added to
Subtraction	decreased by minus, less difference between/of less than, fewer than
Multiplication	of times, multiplied by product of increased/decreased by a factor of (this type can involve both addition or subtraction and multiplication!)
Division	per, a out of ratio of, quotient of percent (divide by 100)
Equal	is, are, was, were, will be gives, yelds sold for

Need of key words betrays inability to comprehend the meaning encoded in the syntax. In other words, it is inability to build mental schemes even in the simplest cases. Those who display this inability are in a trouble much deeper than inability to do mathematics. This is inability to comprehend one's own natural language.

All American eductors, with whom I communicated, treated the method of key words with anger and contempt. However, this method is really useful in dealing with one-step word problems, especially for people with a low mental ability. Let us try to think, is this method really bad. Should we explain meanings of arithmetical operations to children? Yes, we should.

Many educators, including those, whose reputation is beyond any doubt, made tables to help beginners to translate between their native language and algebra.

Polya presents a couple of them [**Polya**], p. 24:

In English	In algebraic language
A farmer has	
a certain number of hens	x
and a certain number of rabbits	y
These animals have fifty heads	$x + y = 50$
and one hundred forty feet	$2x + 4y = 140$

Perelman in one of his books [**Perelman.A**] also teaches the reader the language of algebra on several examples. This is the one of these examples, showing how to translate a famous historical problem into algebra:

In the native language	In the language of algebra
Passer-by, here Diophant's body is buried And numbers can tell you, oh, miracle, how long he lived.	x
One sixth of his life time The beautiful childhood constituted.	$x/6$
Then one twelth part passed, and his chin was covered with hair	$x/12$
One seventh part Diophant spent in a childless marriage.	$x/7$
Five years passed and he was happy to see his first son.	5
Whose fate was to live only half of his father's lifetime.	$x/2$
Deeply distressed, the old man came to the end of his life four years after his son's death.	$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4$

Tell, how old was Diophant when he met his death?

If teaching the language of algebra is necessary, then why are American educators so nervous about the key word method? I think that they have two reasons for that, which may be abbreviated thus: **too late** and **too trivial**.

First reason - age. As we know from Davis, most American students take Algebra I in the 9-th grade when they are 15 years old. At this age to struggle with meanings of common words is late.

Second reason - these explanations deal with single words or isolated phrases, not with syntax. Essentially they simplify the languge – and thereby thought – to a very primitive level.

Instead of forceless anger, American educators should promote usage of multi-step word problems in school curricula, and already in elementary school – then key words will become useless and drop from use.

15. How I taught word problems in an American college

For several years I taught "College Algebra" to freshmen in a Catholic college in Texas. Years ago it had been a female institution and most students still were female. They were at least diligent. Male students were less organized. There were also a few foreign students, all of whom were prepared much better.

The college was proud to provide an opportunity of higher education to those who otherwise perhaps would never get it. However, all my students had successfully graduated from high schools and if these schools had done their job as I understood it, all of my students should be able to solve at least those problems which Russian children solve in elementary school. But this was not the case. I saw my mission in trying to teach my students as many word problems as possible, and for many of them it seemed to be their first experience of doing mathematics beyond isolated arithmetical and algebraic operations. Preparation of students who came to my classes was very poor. Many thought that "X is five less than Y" meant $X = 5 - Y$ and in every semester I had to explain again and again that it means $X = Y - 5$. Generally, I had to teach them to 'translate from English into Algebra and back', and this, as I think, was the most important part of my teaching. Here are a few of those problems which I used.

The college was poor and it was an advantage; nobody pestered me with using calculators or expensive manipulatives. The boards were old-fashionedly large and it was the greatest asset. I came to the classroom with a sufficient supply of chalk and erasers. During class I invited four students at once to the blackboard, asked them to draw lines cutting it into four parts and dictated a problem to all the four at once, only one datum being different. For example, I told them: "Mary has a hundred coins in her piggy bank, some dimes, others quarters. Her total capital is..." While students wrote this, I realized that if all the coins were dimes, Mary would have ten dollars. If some dimes were replaces by quarters, it would increase her capital by a multiple of fifteen cents. So I continued addressing to each student in turn: "thirteen, sixteen, nineteen, twenty two dollars. How many dimes and how many quarters she has ?" Very soon students learned to understand what I meant and spent almost no time writing this down correctly.

I told students that when they are at the blackboard, they are 'teachers' and must care to write clearly so that others could understand them. Whenever a student used a variable, say X, I required her to write down what it meant. For some it was a challenge and this challenge was very useful. Sometimes I requested students to explain their solutions. In these cases I urge them to address their classmates. (They tended to address me or the blackboard.) I also told them that only during tests and quizes they were forbidden to communicate; at all other times they might and should help each other. For example, if a student at the blackboard got confused, her friend came to help her, and this communication also was a valuable experience for both of them.

Solution usually took from five to fifteen munutes. All those who were sitting, were required to choose one version and solve it at the same time. They were willing to do it because they knew that I would allow them to use these notes during tests. I told them: "If somebody at the blackboard makes a mistake, it is your mistake, because you should check each other. I have no time to check every calculation. Even if I see something wrong, I shall not tell." (But actually I left no mistake uncorrected.) From time to time groups of students spontaneously formed in different parts of the room, which discussed these problems. All this was called 'participation' and rewarded by certain points according to my syllabus. In the syllabus I wrote: "It is essential to understand that the study is not a competition. Another student's success is not your failure and your classmate's failure is not your success." Some habits of teachers put students into a competitive position and thereby prevented them from collaboration (for example, grading on the curve). I used various means to do the opposite: at my classes students were friendly towards each other and normally helped each other except when I explicitly forbade this (during tests and quizes).

I also had to correct many bad habits of my students. One of them was a confused and careless manner of writing. Some students, when adding two fractions or doing another arithmetical transformation, cover all the space with crossing lines and intermediate results, so that it becomes impossible to understand what was done, how it was done, whether it was correct and if not, where was the mistake. (This bad habit was partially due to the New Math era, when educators decided that clean and elegant writing was too mechanical.) Another bad habit was 'immediate erasing': as soon as I told a student that her solution written on the blackboard was not correct (sometimes even as soon as I said that I didn't understand it or just asked what it meant), she often immediately erased all of it, so that no further discussion was possible. When all the four versions were solved, I asked if there were questions. Often I made some comments. I explained that one and the same problem could be solved in different ways, for example, using one or two variables or using one or another unit: dollars or cents, hours or minutes.

In the course of this kind of teaching I came to the conclusion that one of the most urgent functions of public mathematical education is **teaching to understand and use intelligently the natural language.** (In the present case English.) You may ask: "Why do we need to teach our students their mother tongue if all of them already know it ?" But there are different ways and levels of knowing one's native language. It takes only a very superficial knowledge to exchange casual greetings: "Hi ! - Hi ! - How are you ? - Just fine. - Take care." It takes much more to comprehend a text describing some system of formal relations. If grammar were taught on a regular basis, it would help a lot, but most of my students seem to have never been taught grammar. (Our ancestors called elementary schools grammar schools. Did they mean something ?) Often speech of my students had a poor coordination with their thinking, sometimes it was generated by authomatisms. Sister Teresa Grabber, a wonderful teacher of remedial algebra in our college, observed once: 'When my students cannot solve a word problem, I discuss with them why they cannot, and we conclude that they cannot read.' I replied: 'Well, you don't mean this literally.' She agreed: 'No, I don't. I mean lack of comprehension.' Regretfully, some other teachers of remedial courses did not teach word problems, so I had to start from scratch.

To avoid misunderstanding, let me note that what I call 'understanding and intelligent use of English' is not the same as what is usually taught in the classes of English. The following example may clarify this. Every semester I had a few exchange students from Japan in my classes. It is remarkable that most of them solved word problems better than most Americans, in spite of poor knowledge of English. To some extent, this was because they had a better mathematical preparation. For example, they did not flounder in simple arithmetics. But this is not all the story. My point is that they seemed to have a better developed ability for mental representations of systems of formal relations, such as are encoded in all word problems.

It is essential to understand that influence of good teachers of mathematics (or of their absence) goes far beyond mathematics. It goes into that unnamed but extremely important nobody's land of general intelligence and culture of thinking which often remains uncovered both by math and English departments. In some families this unnamed land gets explored by various pastimes in such a natural way that when offsprings grow up and become intellectual leaders, they tend to forget that many other people did not have this experience. In result, when asked what to do with public mathematical education, some highly qualified mathematicians come up with well-intended, but unrealistic suggestions based on the tacit assumption that the stage of formal operations is already successfully mastered. They have passed this stage so early and so easily that had no chance to think about its difficulties.

When people lack certain experience, they are often unaware of it. Only a few students understand that they need to develop as persons. Most think that if they take a course, it is to teach them certain practical knowledge and skills, like in a driving school (or just to get a grade). Considering this 'naive practicism' of many students, it helps from time to time to make remarks like 'When you graduate and go into business, you will get bankrupt if you don't understand percentages'. However, as the course goes on, most students feel that they really are learning something more important than practical skills and knowledge. A good course of mathematics should make students understand or at least suspect that there is nothing more useful than a good theory.

I reminded to my students that they must answer the questions which are asked, not just find X, and we had to discuss what these questions meant. For example, many could not figure out independently which quantity is meant when it is asked 'How far away ...', 'How fast ...', or 'How long will it take ...'. I had to teach my students that when they make an equation, they have to choose a certain unit for every quantity. For money it may be dollars or cents and whatever they choose, they have to transform all money data to this unit. For time it is usually hours or minutes, and all time data must be unified also. I also have to remind that there are 60 minutes in an hour, not 100. (Some students, when they need to transform 1/3 hour into minutes, grasp a calculator and come out with 33.3 munutes.) I also have to teach the simplest relations like

$$\text{Speed} \;=\; \frac{\text{Distance}}{\text{Time}}, \quad \text{Time} \;=\; \frac{\text{Distance}}{\text{Speed}} \quad \text{and} \quad \text{Distance} \;=\; \text{Speed} \;\cdot\; \text{Time}$$

Also it was a discovery for most students that if there was no unit for some quantity, they might introduce one in a convenient way and call it as they liked. For example, in "work problems" this ad hoc unit might be called one "Job". Then

$$\text{Rate} \;=\; \frac{\text{Job}}{\text{Time}}, \quad \text{Time} \;=\; \frac{\text{Job}}{\text{Rate}} \quad \text{and} \quad \text{Job} \;=\; \text{Rate} \;\cdot\; \text{Time}$$

In each case I had to write all the three formulas on the blackboard and explain that they are equivalent: most students could not realize this independently. I also had to remind students to write these formulas into their notes; otherwise they forgot to do it and got stuck at the next quiz.

I had to teach my students to organize data. One way to do it was to place them in a table. Let me show how we do it for the following problem.

Problem 37 How much pure water should be added to 10 liters of 60% solution of acid to make a 20% solution of acid ? $\boxed{\textbf{P : acid}}$

Most of my students could not solve such a problem unless I gave them a 'template' to organize the data. One way to do it was to place them in the following table:

	given	added	total
volume in liters	10	X	$10+X$
percentage of acid	$60\% = 0.6$	0%	$20\% = 0.2$
amount of acid	0.6(10)=6	0	$0.2(10+X)$

Since the quantity of acid does not change in the process, we may write an equation

$$6 + 0 = 0.2(10 + X),$$

solving which we obtain the answer: $X = 20$ liters. Let me list some of the mental operations which students need to perform in the course of this solution. (All of them are *not* trivial and at the beginning of the course students make many mistakes.)
- To write appropriate and understandable names for rows and columns, such as 'given', 'added', 'total', 'volume' etc.
- To place data into appropriate boxes.
- To figure out that when two mixtures are poured together, the total volume equals the sum of volumes of ingredients.
- To figure out that when two mixtures containing acid are poured together, the total amount of acid equals the sum of amounts of acid in the ingredients.
- To figure our that pure water contains 0 percent of acid.
- To notice that there are two expressions for the last box, which therefore are equal to each other, that an equation thus emerges, which can be solved to yeld the answer.

Of course, most students could not do all this independently. I had to tell them and there was nothing vicious about it. Some educators, with whom I discussed teaching word problems, scornfully rebuked me for telling students such things. They insisted that to teach students creativity I should somehow move students to discover all that by themselves. But creativity cannot be presented in a flash, like a rabbit from the magician's hat. It needs years and years of intelligent care. Most of my students seemed to have never had a teacher of math who loved or at least understood math. What I could do was to bring some order and structure into their minds. And even this course was too difficult for many students. Those who majored in some science were required to take it, but future teachers of elementary school did not take college algebra. Instead of solving problems they answered questions like “How many diameters are there in a circle ?” or “Given a sequence 5, 10, 20, 40,... According to which rule is it generated ?”). Some of those who took college algebra, tried to reduce solving problems to still more simple rules. And there is nothing mysterious about these difficulties. Remember that algebra is not in our genes; it is in our culture and transmission of culture needs explanations. If explanations have never been given to a person, this person flounders in such problems: puts data into wrong boxes, forgets relations between numbers etc etc. This is not stupidity or inferiority; this is lack of schooling.

It is fashionable among some educators to bash mnemonic devices including tables. But the point is how they are taught. Once a student asked my help in dealing with a problem similar to Problem 37 $\boxed{P: acid}$. I answered: ‘Make a table’. ‘Is it your requirement ?’ - asked the student with irritation. I said - ‘No, but since you say that you are lost, make a table.’ The student started to make a table unwillingly as if she were doing a favor to an old pedant, endeavored to finish it and solved the problem. I said: ‘Now let me tell you something about teaching. You expected me to help you. Did I ?’ She said: ‘Yes, you did.’ ‘But I told you nothing.’ ‘You told me to make a table.’ Teaching is systematic inducing students to self-organization.

Tables are not the only useful form of data representation. Another is graphical representation. For example, when solving “problems on movement”, it helps to use the coordinate system with one or two coordinates to represent movement of objects.

I also had to tell my students the following:

- Write carefully,
- Write every sign in a clear manner,

- Write every equation completely and clearly to make it easy to check,
- Make table carefully,
- Don't write the figures '0', '6' and '8' undistinguishable from each other,
- Don't write the figures '1' and '7' undistinguishable from each other,
- Don't write the letter 'l' just as a vertical bar, undistinguisable from the figure '1'

and many other things which are taken for granted by those who had good teachers in childhood. Is all this mathematics ? The answer, of course, depends on how we define mathematics. But even if this is not mathematics, it needs to be taught, otherwise there will be no mathematics.

One challenge to which I subjected the strongest of my students were 'impossible' problems. Suppose I have four students at the blackboard and dictate: "How much pure water should be added to 10 liters of 60% solution of acid to make: 20%, 40%, 50%, 80% solution of acid ?" All the four students solve the problem in a similar way. Three of them, after all the trials and tribulations, get an acceptable answer, but the fourth answer is negative. I somehow observe that there is something wrong with this. For example, I may ask if somebody can go to a restaurant and order a negative cup of coffee. The student does not know what to do with this. I ask all the class to help her. We check the calculations and see that there is no mistake. Sometimes one of the students suggests the right explanation, sometimes I have to do it. Anyway, I make my students aware that some problems have no answer and that they must be able to make such a conclusion if it is necessary.

Based on this experience I concluded that the purpose of public mathematical education is not teaching mathematics: it is at once much less and much more than that. It is much less because most students will never become professional mathematicians. It is much more because civilization is not in our genes: it takes schooling to transmit it to the next generation. Some of my top priorities as a teacher of mathematics were:

- Make students better understand and use their mother tongue to convey exact information.
- Develop their ability to represent information in ways, which are useful to solve problems.
- Teach students to translate between different modes of representation.
- Improve their manners. (Such as manners of understandable writing, of fruitful communication, including ability to explain, to understand an explanation and to ask the right questions if you don't understand.)

To achieve this, we need definite and exact 'rules of the game': it must be clear what is given, what is asked and how to tell a right answer from a wrong one. Word problems like those described above are most suitable for this.

Part III. International Panorama

16. TIMSS

Around 1995 the Department of Education of USA, helped by analogous offices of many other countries, conducted the Third International Mathematics and Science Study (TIMSS) [**TIMSS**]. Its purpose was to compare the average quality of mathematics and science education at several levels in as many countries as possible. It took several years to organize, describe and present to the public its huge results. Even now there are different opinions about their interpretation. The most shocking result was the relative decline of American students while attending schools: American 4-th graders scored above the world average, 8-th graders were a little below average and 12-th graders were much below average. It is difficult to avoid a conclusion that American children come to school well prepared, but American schools are worse that schools of many other countries. In the 8-th grade (where the number of participants was the greatest) the first several places went to East-Asian countries. Next several places went to European countries and Russia was among them: not worse than France, England or Germany. This is a very good result for a country where most people were illiterate in the beginning of the 20-th century. USA was significantly below these groups. The only Latino-American country mentioned there is Colombia; it is almost at the very bottom of the list. I have no doubt that if Brazil participated, its result would be also very poor.

The disturbing results of TIMSS did not remain unnoticed in USA. On June 9, 1997 Senator Robert Byrd started his speech as follows:

> Mr. President, over the past decade, I have been continually puzzled by our Nation's failure to produce better students despite public concern and despite the billions of Federal dollars which annually are appropriated for various programs intended to aid and improve education. $< \cdots >$ Particularly in mathematics, where our kids will have to be especially skilled, the United States ranks 28th in average mathematics performance according to a study of 8th graders published in 1996. Japan ranked third. [**Byrd**]

Some people thought that since American 4-graders were above world average, American elementary schools are good enough and all that needs to be fixed are middle and especially high schools. This is a naive conclusion. Children's experiences, especially those of good and bad teachers, influence all their lives and certainly influence all their future grades. The relatively good grades of American 4-graders at TIMSS may be attributed to parents even more than to school. (Compare Kenschaft's article.) On the other hand, absence of arithmetics can't help influence all the future intellectual life.

I think that the real situation in American mathematical education is even worse than TIMSS shows because TIMSS followed the anti-theoretical bias of American educators.

By "theory" I don't mean anything too advanced. Let me give an example. When I was in high school, we studied the quadratic function in a rather complete manner. In particular it was obligatory for all Russian schools to teach children to derive the formula for roots of a quadratic equation $ax^2 + bx + c = 0$, to prove that the sum of roots equals $-b/a$ and that their product equals c/a and to use these facts to factorize the trinomial. Most American students whom I ever met were not even aware of most of these facts.

Other immigrants from Russia have similar impressions. Gregory Galperin who always was interested not only in research, but also in teaching, wrote to me soon after starting to teach in America:

> I am very surprised that all (!) the American students know (on some particular examples) how to factor a given quadratic polynomial but do not understand that the numbers inside the brackets are the roots of the equation as well as they do not even suspect what the sum and the product of the roots are equal to. E.g., my graduate students in differential geometry could calculate quite hard integrals to know the area or the length of a curve but were not able to answer the question what the geometric sense of the roots of the quadratic equation $x^2 - 2Hx + K = 0$ is, where H and K are the average and the Gaussian curvature of a surface. However, $H = k_1 + k_2$ and $K = k_1 k_2$ by definition, where k_1 and k_2 are the principal curvatures of the surface, and the students had known these formulas. So I taught them the quadratic equations at their final exam.

As soon as he came to USA, Galperin, in addition to his duties at the university, started to teach high school students and was astonished to hear from them that he taught them "Russian mathematics". In fact, he taught them mathematics rather than so-called new-new math.

Another example: When I was in school, we studied geometry as a deductive system and proved theorems. Although our study was imperfect, the idea was stimulating. In American education this may not even be considered desirable. How do I know this? In the last twenty five years the National Council of Teachers of Mathematics (NCTM), a very powerful organization, published three documents expressing its vision of mathematical education: "Agenda for Action" [**Agenda**] (1980), so-called "standards" [**St.1, St.2, St.3**] (1989) and "Principles and Standards of School Mathematics" or PSSM for short [**PSSM**] (2000). Since PSSM is not intended to substitute "standards" and treats "standards" quite respectfully, criticism of "standards" remains necessary. The two last volumes of the three-volumed "standards" are almost never discussed, so little mathematics they contain. We shall speak only about "curriculum standards", to which we shall hence refer to as "standards" and which constitute the greater part of the first volume [**St.1**]. "Curriculum standards" consists of three parts pertaining to the elementary, middle and high school and we shall concentrate our attention on the two latter parts. I have never taught at the elementary-school level and shall not comment on the elementary-school part of "standards". I only want to say that the elementary-school part seems to be more reasonable than the other two parts. For example, it recommends to increase attention to "mental computation" and "word problems with a variety of structures" (p. 20) with both of which I wholeheartly agree. I would be happy to see similar recommendations in the other two parts, but they are not there.

TIMSS followed American educators in their neglect of theory (but not in their bombastic ambitions; it contained no problems on fractals, non-Euclidean geometry or Student's or chi-square distributions). PSSM also dropped these topics. Algebra and geometry, the two academic subjects most appropriate for study at school, constituted only a smaller part of its "literacy" part and what was presented as "algebra" might be very far from it. For example, the following problem was included into the middle-school part of TIMSS as item 13 and classified as algebra [**TIMSS.item**], p. 76:

Problem 38 These shapes are arranged in a pattern: $\boxed{\textbf{P : TIMSS}}$

$$\bigcirc \triangle \bigcirc \bigcirc \triangle \triangle \bigcirc \bigcirc \bigcirc \triangle \triangle \triangle$$

Which set of shapes is arranged in the same pattern?

A	$\star \square \star \square \star \star \square \square \star \star \square \square$
B	$\square \star \square \square \star \square \square \square \star \square \square \square \square$
C	$\star \square \star \star \square \square \star \star \star \square \square \square$
D	$\square \square \star \star \square \star \square \square \star \star \square \star$

Even if it made sense to include this puzzle into TIMSS (about which I am not sure), it is not even close to algebra! The fact that they call it algebra shows how little of real algebra is there. We cannot even reproach the organizers of TIMSS for their neglect of theory. If TIMSS included more mathematics and less fads, some educators would be still more eager to dismiss its results.

Even the *advanced* part of TIMSS includes very few theoretical topics or problems on proofs: of all its 36 items released on the web only one (K-18) requests to write a proof [**TIMSS.proof**]. Most of the other items require just to choose an answer in a multiple choice format and/or to obtain a numerical answer. Some items contain a requirement "show your work", but the work to show is mostly numerical computations. Even algebraic transformations are almost absent.

There was a special part of TIMSS which allowed to make interesting conclusions: 231 videotapes of eight-grade lessons of mathematics: 100 in Germany, 50 in Japan and 81 in USA. A very interesting book [**Gap:T**] was written based on these videos. The authors write:

When we watched a lesson from another country, we suddenly saw something different. Now we were struck by the similarity among the U.S. lessons and by how different they were from the other country's lesson. When we watched a Japanese lesson, for example, we noticed that the teacher presents a problem to the students without first demonstrating how to solve the problem. We realized that U.S. teachers almost never do this, and now we saw that a feature we hardly noticed before is perhaps one of the most important features of U.S. lessons - that the teacher almost always demonstrates a procedure for solving problems before assigning them to students. This is the value of cross-cultural comparisons. (p. 77)

For example, presenting a problem in Germany sets stage for a rather long devepolment of a solution procedure, a whole-class activity, guided by the teacher. In Japan, presenting a problem sets the stage for students to work, individually or in groups, on developing solution procedures. In the United States, presenting a problem is the context for demonstrating a procedure and sets the stage for students practicing the procedure. (p. 81)

Many U.S. teachers also seem to believe that learning terms and practicing skills is not very exciting. We have watched them trying to jazz up the lesson and increase students' interest in nonmathematical ways: by being entertaining, by interrupting the lesson to talk about other things (last night's local rock concert, for example), or by setting the mathematics problem in a real-life or intriguing context - for example, measuring the circumference of a basketball. Teachers act as if student interest will be generated only by diversions outside of mathematics. (p. 89)

Japanese teachers also act as if mathematics is inherently interesting and

> students will be interested in exploring it by developing new methods for solving problems. They seem less concerned about motivating the topics in nonmathematical ways. (p. 90)

However interesting this book is, it still present an distorted picture as was shown by Alan Siegel [**Siegel**].

17. PISA

In spite of all its deficiences, TIMSS gave a roughly realistic picture of world education. Some educators did not like this result and created another international comparison,called PISA (Programme for International Student Assessment). PISA showed what they wanted it to show: that mathematical education in their countries is not so bad. How did they achieved it? One powerful mean was fuzzification of the test items. We cannot be sure about it, because the true items of PISA are kept in secrecy, but PISA published several items similar to those actually used and a few people commented on them. Let me mention two of them.

One criticism is made by Bastiaan J. Braams and placed on his web site [**Braams**]. In it Braams writes:

> *PISA appears to have been heavily influenced by philosophies of authentic assessment and realistic mathematics education (RME), and offers a valuable perspective on the philosophy and politics of this international mathematics education community.*
>
> *The international PISA web site is* `http://www.pisa.oecd.org/`. *One finds there a description of the testing philosophy and sample questions.* $< \cdots >$ *Key words and phrases: dynamic lifelong learning, real-life situations,*

students' beliefs, self-regulating learning. "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen." $< \cdots >$

For security reasons the actual PISA test is not published, but supposedly representative sample problems and scoring guidelines are provided at [**PISA**].

I will focus here on questions 5 and 6 from the mathematical literacy sample. The questions, described as belonging to the "Big Idea" of space and shape, refer to a drawing of three shapes in the plane. The shapes are labeled (A), (B) and (C). Figure B is close to a circle. Figures A and C look like squid or like an ink-blob: they each have a very jagged edge with lots of indents, and their overall size (diameter) makes it appear that either one might just fit inside or on the circle, figure B. I don't think I'm giving much away if I say that figures A and C have very obviously smaller area and larger circumference than figure B. Here are the questions and scoring guidelines.

Question 5: Shapes

Which of the figures has the largest area? Give explanations to your answer.

Scoring guidelines for Question 5

Score 1: Answers which indicate shape B, supported with plausible reasoning, for example:

• *"B. It doesn't have indents in it which decreases the area. A and C have gaps."*

• *"B, because it's a full circle, and the others are like circles with bits taken out."*

Score 0: Answers which indicate shape B, without plausible support.

$< \cdots >$

"The Circle. It's pretty obvious."

Braams concludes (correctly, as I think):

> The quoted pair of questions illustrates well, I think, some of the degeneracies of present mathematics education research and of the mathematics education reform trends of the past 10-20 years.

Another criticism is made by V. A. Vassiliev and published by Notices of AMS [**Vassiliev**]. Vassiliev refers to the web address
`http://www.pisa.oecd.org/Docs/Download/PISAFrameworkEng.pdf`.
Vassiliev quotes PISA thinkers speaking of three classes of problems. A usual mathematical assignment "Solve the eqution..." is put into the first, lowest class. A rather vague question is put into the second class. But the hit of the show is an item from the third, highest class:

> In a certain country, the national defence budget is $30 million for 1980. The total budget for that year is $500million. The following year the budget is $35 million, while the total budget is $605 million. Inflation during the period covered by the two budgets amounted to 10 per cent.
>
> a) You are invited to give a lecture for a pacifist society. You intend to explain that the defence budget decreases over this period. Explain how you would do it.
>
> b) You are invited to lecture to a military academy. You intend to explain that the defence budget increased over this period. Explain how you would do this.

I also browsed materials of PISA and was disgusted by most of them, but Vassiliev found really a pearl. The quoted assignment essentially is a Freudian slip. It shows the PISA organizers immorality. I completely share Vassiliev's anger and contempt at this item.

In my opinion, the two partially quoted criticisms show the professional and moral level of creators of PISA.

18. Word Problems in Brazil

It is well-known that social contrast in Brazil and Russia are among the greatest in the world: in both countries many people are poor and some are enormously rich. However, with respect to the level of education, Brazil is much more unequal than Russia: there are well-educted people in Brazil, but many are outright illiterate.

It is very important to keep in mind that the two inequalities of Brazil, financial and intellectual, do not coincide. Most students hard working towards academic success, whom I met or taught or adviced in Brazil, were not from rich families. On the contrary, most of these students were short of money. This is important to keep in mind to avoid false conclusions. For example, some people propose to introduce fees to study in universities "because those who study there are rich, so let them pay". I believe that such a measure will be a disaster because it will close university doors for that social stratum, which is most promising for intellectual work: offsprings of families with intellectual traditions, but without money. A telling example is the famous Malba Tahan, whose parents always were short of money, and who became one of the best popularizers of mathematics in the world. Thus a large proportion of future scientists and thinkers study in public schools.This means that advanced programs must be organized for such students.

Also it is important to be aware that existence of private schools and universities in

Brazil does not guarantee existence of competent scientists. According to my observations, most scientific research and teaching for research in Brazil take place in public universities. Offsprings of rich families usually do not go into research; most of them prefer courses which help them to stay rich and manage their richness.

Traditionally, most Brazilians simply were not educated at all. Let us remember the famous experiment conducted several years ago in Brazil with children so poor that they could not afford to attend school and spent their time selling fruits in the streets [**Brazil.kids**]. The experimenters observed that when selling fruits, these children made the necessary arithmetical calculations almost always correctly. However, when these children were invited to a laboratory and asked to make the same arithmetical operations with abstract numbers, they made many mistakes. However, when in the same laboratory they were asked to solve word problems, thus dealing with imaginary fruits, their results were almost as good as in the street.

Now Brazilian government wants all children of school age to be at school. This desire is understandable first of all for humanitarian reasons, and also for practical ones: work force needs to be at least literate and every younster-outcast may be a potential recruit for organized crime. The experiment mentioned above shows that if children of uneducated parents come to school, they probably will need special programs adapted for their lack of abstract thinking, probably with an extensive use of manipulatives. Of course, these programs should not "conserve" these kids at their archaic level; the programs should *develop* their thinking, but to do it carefully, starting where the students are.

On the other hand, as I said, some future scientists and thinkers also attend public schools, and they need programs capable to develop them in the best way. Thus we come to an urgent necessity of tracking or furcation of Brazilian public schools. Putting 100% of children in school without creation of advanced groups will cause a disaster:

Brazil will sink in ignorance.

The situation in Brazilian schools is obscured by fanciful ideas, which please the international community of educrats but only confuse public in Brazil. The corresponding buzz-words are "multi-culturalism" and "ethno-mathematics". The term "multiculturlism"seems to have originated in Brazil. At least, this was written in a small article in the journal "Nova Escola", March 2002, entitled **"To respect everybody's culture"**:

> The term ethnomatematics was proposed in 1975 by Ubiratan D'Ambrosio *(foto below)* to describe mathematical practices of cultural groups, would it be a society, a community, a religious group or a professional class. These practices are systems of symbols, spacial organization, techniques of construction, methods of calculation, systems of measures, strategies of deduction and resolution of problems and any other action, which can be converted into formal representations. Originally, ethnomatematics started with a historiografic vision of cultures of the past. "How did colonizator dominate the colonized? Imposing a new language, a new religion, a new mathematics", sais D'Ambrosio. Indians. lapões, african tribes, all had (and have) their own way to analyse and quantify, which was forbidden. "It is very arrogant to imagine that the only ones who thought and had logic were mediterranian people." Today, with more information, this knowledge reappears. "This is an impressive richness, which must be considered and respected", the professor observes. Mixing all these contributions, we can come to a wisdom more universal and democratic, without impositions. According D'Ambrosio, the principle is valid also for the microuniverse of classroom. A student from favela, sons of artists or engineers, all have some informal way of using mathematics. None teacher may act as a colonizator. "To open one's mind and

> to recognize the reality of the group is a precious chance which we have to establish unity with the student", he teaches.

The same text is put on Ubiratan D'Ambrosio's home page. This text is kind of narcotic for students from poor families. It flatters them and gives them some pleasant dreams, but the awakening will be painful. It is a reality that children of poor and uneducated parents are badly prepared for intellectual studies. It is a sad reality, but still reality. It is possible to improve this reality only having first admitted its existence. D'Ambrosio simply negates this reality, he invites poor families to an illusory world where their children are prepared for school as well as children of middle class; but in the final analysis their life will be hard.

The multicultural movement have been frowned upon by most serious thinkers. In particular, Diane Ravitch wrote about it:

> The main effect of public brawls about multiculturalism was to divert attention from the urgent need to improve the quality of teaching and learning, a subject that was never as newsworthy as confrontational battles over race and ethnicity. [**Ravitch**], p. 420

As I said in section

19. Brazil

poverty of teachers is the hardest problem in public education of many a country, including Russia and Brazil. In Brazil teachers' salaries grow, but still are miserble.

On October 17, 2000 the newspaper *Jornal do Brasil* published article in which the Ministry of Education proudly announced growth of school teachers' salaries. . In the article "Balanço é positivo" it claimed:

> From December of 1997 to June of 2000, salaries of teachers with complete fundamental education grew from R$ 165 to R$ 324. Those who have a complete magister course, gained most, their salaries grew from R$ 288 to R$ 504. For those who have incomplete fundamental education, the salary grew from R$ 177 to R$ 295.

The greatest salary mentioned here is R$ 504. I can tell you by experience that it is very hard even to survive on 500 reals per month. It is a misery. I have plain tastes, but still I spend around this amount only on food for myself.

For a long time only a small fraction of Brazilians were educated. In the recent years the Brazilian government decided to introduce education for all. In this connection,

In recent years the Brazilian Ministry of Education published national standards for grades 1-4 [**St.Br.1-4**] and grades 5-8 [**St.Br.1-4**].

Let us first speak about the 1-4 standards [**St.Br.1-4**]. FIrst, information on its cover is strange. It is labeled as 3-d volume. I have never seen 1-st or 2-d volume. Also it does not say explicitly, to which grades it pertains. One has to look inside to conclude that it is about 1-4 grades. However, there is no hint, what to do at each grade, which is especially mysterious since practically all problems in the book are one-step. Examples:

Problem 39 Marta wants to buy three packs of chocolate. Every pack costs R$ 8,00. How much will she pay for three packs? (p. 110) $\boxed{\textbf{P}:\textbf{Marta1}}$

Problem 40 Marta payed R$ 24,00 for 3 packs of chocolate. What is price of every pack? (p. 110) $\boxed{\textbf{P}:\textbf{Marta2}}$

Problem 41 Marta spent R$ 24,00 to buy several packs of chocolate. Each pack costs

R$ 3,00. How many pcks of chocolate did she buy? (p. 110) **P : Marta3**

It was a good idea to place these three problems together. But their level corresponds roughly to the second grade. Where are problems for the fourth grade?

Problem 42 Area of a retangular figure is 54 cm^2. If one side is 6 cm, how long is the other side? (p. 111) **P : side**

Almost all problems in the book are one-step and deal only with positive integer numbers within 100. The following problems is the only exception:

Problem 43 Two pineapples cost R$2,50. How much shall I have to pay for 4 pineapples? (p. 110) **P : pineapple**

Its solution can be written this way:
a) 2,50 / 2 = 1,25.
b) 1,25 $\times$ 4 = 5.
However, the numbers are so small that it is easy to understand that 4 is twice 2, so the problem can be solved in one step: 2,50 $\times$ 2 = 5.

The following problem also is special:

Problem 44 At a party, it was possible to form 12 different pairs todnce. If there were 3 girls and all danced, how many boys were there? (p. ?) **P : dance**

Formally speaking, the solution consists of one operation: $12 \div 3 = 4$ boys. However, every competent educator would notice that this problem does not belong here. It belongs to the area of combinatorics, which is much more sophisticated.

Since publication of this book, two Brazilian ministers of education changed their places. I don't know what the present minister thinks of these problems. To me, these problems are acceptable for the first grade, perhaps second, but not fourth.

Also I am disgusted by the general conceited tone of this book. It is written as if a victory were made and never admits that children at this age might do more. I can believe that what takes place in some provincial schools is even worse, but this is regrettable and the ministry should admit it.

It is interesting that some authors of Brazilian textbooks do not follow theministry's recommendations. One of the most popular Brazilian texbooks for the 4-th grade contains more sophisticated problems, for example this:

Problem 45 How many liters of water remain in a tank, which is 12 m long, 6 m wide and 5 m high if 20% of its capacity is spent? [**Marcha**], p. 265 **P : water**

The number of steps in its solution is from 3 to 5 depending on what we consider a step. The first two steps allow to calculate the volume: 12 x 6 x 5 = 360 litros. The third and fourth steps: 20 percent of 360 is 360 x 20/100 = 72 litros. The last step: 360 - 72 = 288 litros. All the other problemsin the book are much more simple, but many of them need at least two steps, so the book is better than the ministry's recommendations.

Now about the 5-8 stanhdrds [**St.Br.5-8**].

Let us look at some problems in this book. On p. 108 we find this:

Problem 46 A year ago Carlos'height was 1,57 m. During the last year he grew 0,12 m. What is Carlos'height now?

This problem can be solved in one step, which makes a strange impression. What dod the authors want to tell the public: that such problems should be solved in 5-8 grades?

On p. 109 we find a more complicated problem:

Problem 47 A building has two water tanks, each with capacity 5000 liters. One of them is 1/4 full and the other contains three times more water. How many liters of water the building has now?

Formally speaking, this is a three-step problem if we solve it this way: a) $5000/4 = 1250$. b) $1250 \times 3 = 3750$. c) $1250 + 3750 = 5000$.

However, it is easy to reckon mentally that if we take a tankfull of water for a unit, the first tank has 1/4 unit, the second has 3/4 unit, so together they have $1/4 + 3/4 = 1$ unit, that is 5000 liters. Solved this way, it becomes a two-step problem. And it is the hardest problem in the book!

On p. 81 the book recommends to "resolve situations-problems involving natural, integer, racional and irrational numbers, amplifying and solifying the meanings of addition, subtraction multiplication, division, potentiation and taking roots." It never mentions solving problems that need more than one operation. Also it proposes to "resolve situations-problems using equations and inequalities of the first degree, including involved procedures." However, I found none problem involving irrational numbers or potentiation or taking roots or inequalities.

20. Different Countries, Similar Proposals

Russia and Brazil are different countries, but have something in common. In particular, both have Ministry of Education. More than that, both ministries lately made similar proposals – to merge final exams in high schools and entrance examintions in all universities into some universal union. I shall call the Russian version "UFE", an abbreviation from "Universal Federal Examination". The Brazilian version is called

"ENEM", which means "Exame Nacional do Ensino Médio". In both countries practically all universities rejected this measure. In Russia they made it in a more dramatic form, which allows us to speak about an educational war in Russia. In particulr, in June 2004 a large group of Russian intellectuals sent an open letter to President V. V. Putin entited

"No"– to destructive experiments in education [No]

It was signed by 420 noted scolars including 86 members of Russian Academy of Sciences, 253 university professors and many outstanding teachers and educators.

Brazilians did not make so much noise, but essentially ignored a similar proposal from their ministry of education.

I support Russian and Brazilian scolars in their rejection of these impossible examinations. I completely agree with Russian scolars when they write in their letter:

> Combination of school finals and university entrance examinations is impossible in principle. The aims of general school education and professional university education are different in principle.

I think that this is even more true for Brazil than Russia, because educational contrasts are sharper in Brazil.

21. Influence of American Fuzzy Math in Israel

Here is a very telling example of influence of the American Fuzzification outside USA. Ron Aharoni is a professor at department of mathematics of Technion, Israel. He had been interested in education for a long time, but probably had not expected that he would have to defend it in such a dramatic form. This is a quote from his recent article [**Aharoni**] posted at his web site:

> Mathematical education in Israel is at a low ebb. In 1964 Israel took first place in the international tests in arithmetics for elementary school students. In 1999 it was in the 28-th place, among 38 nations, between Thailand and Tunisia. A recurring complaint from secondary school teachers is that students arrive from elementary school with very scarce knowledge of fractions.
>
> It is hard not to relate the deterioration to the changeover which took place at the end of the 70-s. Almost overnight the textbooks were changed then in most Israeli schools, to books following the so called "structural approach". The developers of the method ascribe the failure to other factors, mainly the weakness of teachers. Clearly, had Israel gone from 28-th place to first they would have ascribed the success to the method.
>
> The Israeli "structural approach" followed in spirit the "New Math" reform that occured in the US in the 1960-s, and was already abandoned there by the time it reached the shores of Israel.

Further the article says:

> As a possible remedy for the grim situation, the ministry of education decided to replace the elementary school curriculum. A committee of 10 people was formed to write a new curriculum. As its head was appointed Prof. Pearla Nesher, who had been the leader of the "structural approach" reform. $< \cdots >$
>
> The proposed curriculum follows very closely yet another reform that occured in the United States – the 1989 "Standards". $< \cdots >$ The main change in

> the proposed curriculum is a drastic cut in material. Basic topics are left out. Very little is left of the teaching of fractions; the standard algorithms for addition and multiplication are not included, as is also long division. $< \cdots >$
>
> Where do 10 people find the courage to make such drastic decisions, which may affect the scientific and industrial future of the entire country? Probably from the thought that what is good for the US must also be good for us.
>
> Unfortunately, the premise is wrong. The "Standards" reform has not done much good for American education. In fact, its results can be described as total failure. The American educational scene is still in turmoil, with the "math wars" that followed its adoption. California, which was first to adopt it, abandoned it in 1998. It was replaced there by a traditional, subject matter oriented curriculum, leaving didactics to the teachers. **If Israel wishes to follow the steps of the US, it could take this curriculum as a model.**

The bulk of Aharoni's article is a critical examination of the proposed curriculum, with a conclusion that this curriculum should be halted and completely rethought by a panel of referees "including mathematicians, computer scientists, scientists from other areas and leading figures from the Hi-Tech industry". I think that a similar suggestion is good for any country.

Part IV. American "standards" of school mathemtics

22. What NCTM calls "standards"

In the last twenty five years the National Council of Teachers of Mathematics (NCTM), a very powerful organization, published three documents expressing its vision of mathematical education: "Agenda for Action" [**Agenda**] (1980), a trilogy of so-called "standards" [**St.1, St.2, St.3**] (1989-1995) and "Principles and Standards of School Mathematics" or PSSM for short [**PSSM**] (2000).

All of these documents are vague, loose and desorganized. The list of "Recommendations for School Mathematics of the 1980s" on p. 1 of Agenda is a bizarre mixture of suggestions, some of which look sound (at least, at first sight), some are unclear and some are outright unsound.

The first one is "problem solving be the focus of school mathematics in the 1980s", which seemed sound to me at first. Hiwever, Frank Allen, a long-time insider of NCTM, read between the lines of this suggestion: "Let us not teach theory". The subsequent developments showed that at this Allen was right. However, even Allen, with all his experience, naively believed the explicit message: let us teach to solve problems. The subsequent years showed that as soon as these educators evaded the Scylla of theory, they started to avoid the Charybdis of problems. They claimed that traditional problems were too bad for them and proclaimed creation of new, much better problems, which however never materialized.

The second is "basic skills in mathematics be defined to encompass more than computational facility", which is obscure until somebody clarifies what exactly beyond computational facility should be encompassed. This clarification never came.

The third is "mathematics programs take full advantage of the power of calculators and computers at all grade levels", which is a call in a wrong direction as we had plenty of chances to see in the subsequent years.

Closer to our theme, "Agenda" suggested that

> *The definition of problem solving should not be limited to the conventional 'word problem' mode.* [**Agenda**]*, p. 3*

This inarticulate suggestion is a telling example of degradation of American educatiobal language and thought. When I read classics of American education, I might disagree but I always understood what they meant. However, in the present case it is only clear that there has been something wrong with word problems in America already in 1980, but what – the authors cannot explain because they never made it clear for themselves. Of course, no meaningful action could be undertaken based on that immature suggestion.

What about the 1989-1995 “standards”, the two last volumes are almost never discussed, so little mathematics or anything meaningful they contain. Let me speak only about “curriculum standards”, to which I shall refer to as “standards” and which constitute the greater part of the first volume [**St.1**].

“Curriculum standards” consists of three parts pertaining to the elementary, middle and high school and I shall concentrate my attention on the two latter parts. I have never taught at the elementary-school level and shall not comment on the elementary-school part of “standards”. I only want to say that the elementary-school part seems to be more reasonable than the other two parts. For example, it recommends to increase attention to “mental computation” and “word problems with a variety of structures” (p. 20) with both of which I wholeheartly agree. I would be happy to see similar recommendations in the other two parts, but they are not there.

First of all, I must say that “standards” is a difficult reading for a mathematician who has got used to expect exact meanings. It is written in a very fancyful manner, many words have strange meanings or seem to have no definite meanings at all. For example, chapters are called “standards”, which gives impression that there are some standards there. (Chapters of PSSM are also called “standards”, which leads to the same confusion.) But if you apply effort and concentrate, you notice that this looseness is not only in *how* this text is written, but also in what it *recommends*. This document is written by people, for whom all mathematics is but a disordered collection of interchangeable appendices to their vague generalities. (The same is true of PSSM.)

The "standards" contain several reasonable problems (along with several unreasonable ones), but all of them are torn out of their natural mathematical context. We, mathematicians, have got used that mathematics is structured around its content and that all its statements are connected, and that the logical inference is the most important connection. The "standards" carefully avoid to speak about logical connections. Each part contains a chapter named "Mathematics as reasoning", but all the three chapters contain very little reasoning. Through its long history mathematics has collected many important but elementary proofs to fill a chapter with such a title, for example many beautiful geometrical theorems including the famous Pythagorean theorem. Also the famous proofs that $\sqrt{2}$ is irrational, that when a rational number is turned into a decimal fraction, it is periodic and vice versa and that there are infinitely many prime numbers. I would also put there criteria of divisibility by 3 and 9 (and 11) and some of the most elegant proofs by the method of mathematical induction and some of the most thrilling fallacies and paradoxes. As a university teacher, I certainly want my students to understand all this and solve related problems. Is it possible to study proofs in high school? Theoretically, yes. We are not aware of any biological or psychological law to forbid this.

According to Piaget's observations, most children in the better schools of Geneva around 11-12 years old reach and around 14-15 years old complete the stage of development when they master formal operations. (See e.g. [**Piaget**], p. 161.) We may expect similar developments to take place in other industrial countries. Mathematical education should use this opportunity and we have a proof by experience that this is possible: some schools successfully teach such things [**Shen**]. Remember also mathematical olympiads where teenagers solve problems which need rigorous reasoning. However, this theoretical possibility is realized in practice only when there are favorable conditions, first of all good teachers who know and love their subject.

Now about the three chapters "Mathematics as reasoning" in the "standards": none of the famous facts listed above is there. What is there? The high-school chapter starts with tampering with calculator. If you don't believe me, look by yourself. What about the Pythagorean theorem, it is mentioned in the "standards" on pp. 113-114 with a well-known picture, which can be used to prove it, but it is only proposed to use this picture to "discover this relationship through exploration". The possibility to prove this important theorem is not even mentioned and the very idea of proof is avoided throughout the document.

The authors of "standards" think that they write about mathematics, but all they write is "out of focus", like a bad photo. They start with some generalities, some of which may seem reasonable at first, but do not especially care which concrete mathematical content (if any) they use to illustrate them. This makes a sharp contrast with Russian programs which contained no vague generalities at all, just detailed lists of topics, which were very difficult to misinterpret. In those few cases when vague generalities were included into Russian programs, they always led to negative consequences. For example, a program in physics for university entrance examinations recommended to distinguish between genuine understanding and being coached, which immediately moved some examiners to frown upon applicants who knew too much and therefore were perceived as "coached". Generally I am convinced that there is nothing worse than include words like "understanding" into documents called "standards". Modern American educators do exactly this: PSSM abound in recommendations to achieve "deep understanding" but fail to specify what exactly should be understood.

Another unhappy feature of "standards" is complete absence of connections with other sciences. Isolation of subjects from each other, notably between mathematics and physics, is a chronic disease of American education. What about physics, many American school students simply never take it and nobody tells them that they miss something important. Once, teaching a course of calculus, I solved a mechanical problem and then said: “The same result can be obtained from the law of conservation of energy.” Silence. I asked: “Who has ever heard about the law of conservation of energy?” There was one foreign student in the group (from Taiwan) and he was the only one to raise his hand. In Russia every school student was required to take all the main courses including mathematics and physics and the courses of mathematics and physics were always stronly correlated. The course of physics was full of problems that needed algebra, geometry and trigonometry to solve and the course of mathematics included many problems with physical content.

Pages 126-127 of [**St.1**], are devoted to the ‘Summary of changes in content and emphasis in 9-12 mathematics’. Page 126 is devoted to *topics to receive increased attention* and the first topic is **The use of real-world problems to motivate and apply theory.** Page 127 is devoted to *topics to receive decreased attention.* and the first topic is **Word problems by type, such as coin, digit, and work.**

When I read this first time, I was confused. I thought: ‘Do coins exist in real life ? It seems that they do. Then why should they receive decreased attention if real-world problems should receive increased attention ? And what about work? It also seems to exist in real life. And if coin problems should receive decreased attention, what to do with problems about paper money or checks or money orders ?’ I shared my doubts with some American educators and they kindly explained to me (in private, not in public) that the phrase ‘by type’ meant the widely used but uncreative manner of teaching, when the teacher starts by describing in detail a certain method and then she and her students solve many almost identical problems using exactly this method. Taught in this way students can solve problems of a certain type, but often get lost when given even a slightly different problem. I understand this explanation, but I still do not understand what is written in [**St.1**]. If 'by type' means some way or manner of teaching, then how can it be listed as 'topic'? And what do coin, digit and work here? The more problems, such as coin, digit or work, are purged from the curriculum, the more uniform and monotonic become those few problems that remain. The main problem is not with the word problems, but with the poor preparation of American teachers. Polya quoted one prospective teacher say, "The mathematics department offers us tough steak which we cannot chew and the school of education vapid soup with no meat in it". H. Wu also explained this very well:

> The Standards should be more careful in suggesting what topics to omit or de-emphasize, and even more careful in the exact phrasing of these suggestions. $< \cdots >$
> On p.127, it is suggested that "Two-column proofs should receive decreased attention". The phrasing carries the implication that there is something wrong with two-column proofs per se. Of course this is absolutely false: this is an excellent vehicle to guide the students' first steps in trying to write a proof. Two-column proofs get such a bad rap because most teachers do not understand proofs, with the consequence that they inevitably abuse two-column proofs and make them a liability in mathematical education. Thus to these people, the recommendation that "Two-column proofs should receive decreased attention" (without a carefully worded explanation to go with it) carries an automatic invitation to do without all proofs. Lest this statement be taken as an unwarranted exaggeration, may I point out the recent appearance of geometry texts which do essentially no proofs but only "experimental geometry". $< \cdots >$
> Also, on p. 127, it is suggested that "Word problems by type, such as coin, digit, and work should receive decreased attention". Pretty much the same

> comment as above again applies: there is nothing wrong with coin, digit or work problems. Some of these are very good problems. What is wrong is that in the hands of unqualified teachers, these problems become meaningless drills. < · · · > What needs fixing is the teacher qualification problem. NCTM should find (diplomatic) ways to express this fact correctly. The present recommendation concerning ”problems of type” is misleading at best. [**Wu.standards**], p. 1-2

In fact, the phrase “problems by type” had been used in American educational literature already several decades ago. Ernst R. Breslich, an undeservedly forgotten educator, wrote half a century ago:

> Pupils lack sufficient imagination to picture the problem situations. They do not have the ability to connect these situations with their own experiences. Textbooks usually try to develop the ability by grouping problems according to types. Thus, problems are classified as motion problems, digit problems, physics problems, and mixture problems. These and many other types are given intensive treatment. One of the first questions the pupil is told to ask himself is: What type of problem is it? As soon as the classification is made, he is expected to choose a technique especially devised for solving problems of that type.
>
> The method of teaching problems only by type has its disadvantages. Problems in everyday life do not occur in classified sets. The method is unnatural and places more emphasis on teaching “cases” than on developing ability to solve problems that do not come under any of the types taught, or that appear in sets in which the types are mixed. < · · · >
>
> The best practice for the teacher to follow is to group problems as frequently in mixed sets as by types. This retains the advantages of the case method and

> eliminates the disadvantages that may arise from it. [**Breslich**], pp. 188-189

Thus, according to Breslich (and to his contemporaries) “by type” is not a quality of problems themselves, but a certain order of their presentation. An individual problem cannot be “by type”; only a large quantity of problems can be (or not be) by type. Breslich’s recommendation is also quite understandable (and moderate as is usual for him): sometimes present problems by type, sometimes mix them. The “standards”, having recommended to decrease attention to “problems by type”, never said, which order to use instead of it. On the contrary, on the opposite page they recommended to increase attention to “real-world problems”, which made a lot of people think that problems by type are a special kind of problems (such as coin, digit, work), which should be excluded (not mixed as Breslich recommended) to give place to a new better kind of problems. Even people, who should be well-informed, seem to get this distorted message. For example, PSSM [**PSSM**] contains no coin, digit or work problems. But where is that vaunted new kind of problems? It did not materialize. PSSM contains very few problems of any kind.

The high-school part of “standards” contains a list of topics to increase attention, where the first place is given to “the use of real-world problems to motivate and apply theory” (p. 126). What is a “real-world problem”?

Browsing through “standards”, I found quite a few statements about these mysterious critters. On p. 76 (middle-school part) it is said:

> The nonroutine problem situations envisioned in these standards are much broader in scope and substance than isolated puzzle problems. They are also very different from traditional word problems, which provide contexts for using particular formulas or algorithms but do not offer opportunities for true problem solving.

What? What did they say about traditional word problems? What a nonsence! With their narrow experience the authors pretend to set standards! Are they aware of the rich resourses of excellent traditional word problems around the world? Let us read further:

> Real-world problems are not ready-made exercises with easily processed procedures and numbers. Situations that allow students to experience problems with "messy" numbers or too much or not enough informations or that have multiple solutions, each with different consequences, will better prepare them to solve problems they are likely to encounter in their daily lives.

Pay attention that the author uses future tense. This means that he or she has never actually used such problems in teaching and never observed influence of this usage on his or her students' daily lives. He or she has not even invented such problems because he or she does not present any of them. Nevertheless, he or she is quite sure that these hypothetized problems will benefit students. What a self-assurance!

After such a pompous promise it would be very appropriate to give several examples of these magic problems. Indeed, we find a problem on the same page, just below the quoted statement. Here it is:

Problem 48 Maria used her calculator to explore this problem: Select five digits to form a two-digit and a three-digit number so that their product is the largest possible. Then find the arrangement that gives the smallest product. **P** : **product**

This is a good problem, although rather difficult for regular school because having guessed the answer, Maria needs to prove it. But the author never mentions the necessity of proof. What does the author expect of calculator's usage here? It can help to do the multiplications, but it cannot help to prove. It seems that the author expects Maria to try several cases, to choose that one which provides the greatest product and to declare that it is the answer. But what if the right choice never happened to come to her mind? This is very bad pedagogics. Also let us notice that Maria is expected only to "explore" this problem rather than to solve it. According to my vision, exploration is the first stage towards a complete solution. Do the authors expect Maria ever to attain a complete solution? Do they want children to **solve** problems or just to tamper for a while?

But let us return to our main concern: so-called "real-world problems". Notice that this problem has **none** of the qualities attributed to these mysterious critters on the same page: there is neither too much nor not enough information and there are no multiple solutions, each with different consequences.

One colleague noticed that the book still contains some problems described on page 76. Indeed, there are, but in another document. Here is one of them:

Problem 49 You have 10 items to purchase at a grocery store. Six people are waiting in the express lane (10 items or fewer). Lane 1 has one person waiting, and lane 3 has two people waiting. The other lanes are closed. What check-out line should you join? [**St.1**], p. 212 $\boxed{\textbf{P} : \textbf{shop}}$

I have never read any report about usage of this problem. Also I have never read any solution of this problem. Irresponsibility again!

What about problems with too much or not enough informations, they attract much attention in Europe lately, but European scolars want children to treat them critically and in many cases **to refuse to solve them!** Take for example that famous problem, after which Stella Baruk named her book [**Baruk**]. In the late seventies, the following problem was given to 97 second and third graders of primary school in France:

Problem 50 There are 26 sheep and 10 goats on a ship. How old is the captain? $\boxed{\textbf{P : captain}}$ [**Baruk**], p. 25

76 children (out of 97) presented a numerical answer obtained by tampering with the given numbers. For instance, they might add the numbers and declare that the captain was 36 years old. Educators of several European countries (France, Germany, Switzerland, Poland) are very preoccupied by the fact that children "solve" unsolvable problems. The European educators would be very pleased if children refused to solve such problems with a comment like "It cannot be solved". The European educators are quite right. But the same is true of what the "Standards" call "real-world problems". The most sound reaction to the problem 49 $\boxed{P : shop}$ is "I don't know". But what a grade will an American student get after that?

There is only one problem explicitly called "real-world" in the whole volume of "standards". Here it is [**St.1**], p. 139:

Problem 51 "*Real-world problem situation.* In a two-player game, one point is awarded at each toss of a fair coin. The player who first attains n points wins a pizza. Players A and B commence play; however, the game is interrupted at a point at which A and B have unequal scores. How should the pizza be divided fairly? (The intuitive division, that A should receive an amount in proportion to A's score divided by the sum of A's score and B's score has been determined to be inequitable.)"
This is followed by "Problem formulation":
"Consider the situation with the following data: The winning score is $n = 10$; when the interruption occurs, the score is A:B=8:7. The pizza will be divided in proportion to each player's probability of winning the game."

This "problem formulation" is equivalent to that described by Pascal in his letter to Fermat on August 24, 1654 and it is used in introductory textbooks of probability, e.g. in the excellent books by Chung [**Chung**], p. 26-28 and Snell [**Snell**], p. 3-5. Both Chung and Snell refer to Pascal's letter and provide interesting historical background. On the other hand, the "standards" completely omits all historical details, which makes all the situation far-fetched. I asked many sympatizers of "standards" to refer to any of their acquaintances who ever played this game and received none positive answer. So in which sense it is "real-world"? If the author were serious about real-world fairness or equity, he or she might recommend to divide pizza equally or give a bigger piece to the more hungry player, but certainly *not* to divide food by gambling. Administrators of orphanages would be horrified by this idea; they care that each pupil consumes all the food she needs for her health and not gambles it away. Further, what is the mathematical meaning of the word "inequitable"? If there is no such meaning, then how could it be "determined" that some division is "inequitable"? What does the word "determined" mean? The author is trying to herd the readership towards the well-known solution without explaining in which sense this solution is correct, which is anti-mathematical, even anti-rational. Further, the author pretends that the requirement to divide the pizza in proportion to each player's probability of winning the game appears only in the "Problem formulation", but in fact it is implicitly present from the very beginning, because otherwise it can not be "determined" that to share the pizza in proportion of the players' scores is "inequitable". This assumption is implicit, but unstated, which is anti-pedagogical. Even leaving all this aside, educational value of this example is very doubtful. It may seem to be an advantage to include probability into the high school curriculum, but in fact the "standards" avoid any theory: it just tampers with one numerical example, which is a step backwards by comparison with any reasonable version of the traditional curriculum.

The very idea of school is that it is *organized* so that children do not waste their time: they are taught according to a carefully prepared curriculum and solve carefully designed, selected, edited and field-tested problems. The authors want to disrupt this efficiency and present this as an achivement!

Let me emphasize that my criticism is not directed against problems which have more than one or none answer or problems with missing, surplus, irrelevant or contradictory data or other special kinds of problems. All of them have a place in education and teachers may use them for special dramatic effects. (See how I used them in the section 15 based on [**Toom.How**]) What I am against is an apology of irresponsibility, an idea that teachers, authors of textbooks and educational officials should not carefully prepare the learning process. They should, and every special situation should be planned and rehearsed in advance, like special effects in theater or circus.

In fact, the description of "real-world" problems quoted above explains the authors' inability to provide examples of them. Since these perishable fruits are not ready-made, they cannot be found in a book, because any book is ready-made. Although the "standards" is an exceptionally careless document, it was revised once before publication (according to its preface), so it cannot contain "real-world problems" because a problem revised at least once becomes ready-made and cannot be real-world any more. Real-world problems just happen in the actual course of daily life, at least this is what I conclude from the "standards". What the authors lose from sight is that a mathematician, used to concentrate on abstractions, is not the best person to deal with such events. An experienced handyman or family doctor or life-saver or police officer would be much more heplful. It may be a good idea to teach children to cope with emergency situations, but it is not mathematics.

Another relevant statement can be found on page 157 of "standards" (high-school part): "Prior to the work of the ancient Greeks (e.g. Thales, Pythagoras), geometric ideas were tied directly to the solution of real-world problems." Thus problems solved thousands years ago, when there was no theory, were "real-world problems". But page 126 recommends to use them to motivate and apply theory. How can problems posed and solved in the absense of theory motivate and apply theory? And how would a peasant of Ancient Egypt react if the government official in charge of measuring area of his farm would obtain several answers, each with different consequences? Wouldn't such an official be hastily removed, lest he might provoke a rebellion?

Don't ask such questions, because you will never get an answer. Throughout all these years of bitter arguments about what this or that phrase of "standards" really meant, its authors never interfered with explanations. It looks like they wrote the "standards" in such a somnambulic state of mind that afterwards could not explain rationally what did they mean. Nevertheless this irresponsible document is awed by American educrats. The next vision of NCTM, published ten years later, writes about the "standards": "Since their release, they have given focus, coherence, and new ideas to efforts to improve mathematics education [**PSSM**], p. ix.

Another bizarre consequence of the same recommendation: some people guessed that "real-world problems" are those which mention brand names. Some textbooks included problems like this: "The best-selling packaged cookie in the world is the Oreo cookie. The diameter of an Oreo cookie is 1.75 inches. Express the diameter of an Oreo cookie as a fraction in simplest form" [**brand**]. This produced a wave of criticisms, to which the publishing house representative Jack Witmer answered: "Time and again, teachers tell us that the use of real-world examples is effective in engaging students' interest and in enhancing the learning process." [**Witmer**] This sounds unconvincing, but what to do instead? Nobody knows.

It must be said that the "standards" have a wide popularity among American educators. I think it is because the vague feelings of the authors are close to the vague feelings of their audience. All of them feel that their teaching is too rigid, mechanical, uninspired and want to make it more flexible, more human, but they are not competent enough to keep mathematics on this way: as they move towards more human approach, they lose mathematics.

What was the position of American mathematicians towards the "standards"? This is another mystery. The preface to the "standards" declares (p. vi): "The following mathematical science organizations join with the National Council of Teachers of Mathematics in promoting the vision of school mathematics described in the Curriculum and Evaluation Standards for School Mathematics", followed by an impressive list including the American Mathematical Society (AMS). What does this "promoting the vision" mean? Neither AMS nor NCTM ever made a public statement about it. When the "standards" started to be implemented in some classrooms and mathematicians became aware of what was going on there, they became horrified and only then, probably, some of them looked attentively under the cover of "standards". This was not easy, because the "standards" is written in such a vague manner as to make it especially difficult to read for mathematicians. However, some ones succeeded to make some sense out of it. Notices of AMS published several letters urging AMS to withdraw its endorsement (whatever it meant), but there was no comment from the headquarters of AMS.

Since the "standards" seemed to be endorsed by so many highly scholar organizations, it is no wonder that many teachers declared that they teach according to these "standards". When these teachers were asked why did they think so, most of them answered: "Because my students use calculators instead of doing paper-and-pencil calculations". This was said with pride because the "standards" really make impression that it is urgent to increase attention to use of technology, including calculators, and to decrease attention to paper-and-pencil calculations. The "standards" turned calculator into a symbol of prestige and teachers whose students did not use it, started to feel obsolete and inadequate. There were bombastic promises that usage of calculator would release children's time to acquire "high-order thinking skills", but in fact the opposite was observed. Many university teachers complained that their students cannot do simple calculations.

Stiegler and Hiebert tell about the following episode:

> When we examined the places in the video that teachers referred to as examples of reform, we saw a disturbing confirmation of the suspicion we voiced in Chapter 6 – that reform teaching, as interpreted by some teachers, might actually be worse than what they were doing previously in their classrooms.
>
> One teacher, for example, pointed to her use of calculators as an instance of reform in her classroom. True, NCTM recommends that calculators be introduced early in the curriculum, because, among other reasons, they can save computing time so students can focus their attention on problem solving and conceptual understanding. But this was not the way calculators were being used in this particular teacher's classroom. Midway through the solution of a simple problem, the class needed the answer to the problem $1-4$. "Take out your calculators," the teacher said. "Now, follow along with me. Push the one. Push the minus sign. Push the four. Now push the equals sign. What do you get?" The calculator, in this case, was a diversion, and accomplished little on behalf of students' mathematical understanding. [**Gap:T**], p.106

The following was sent to an e-mail list by Lawrence Braden, a well-known teacher, one of the authors of [**Fordham.RB**]:

> The 'do not teach the child fractions except by calculator' is not a rare thing these days. What would be condemned as heresy twenty years ago is now accepted orthodoxy in many circles. One state prides itself in *not* requiring students to know how to add one third to one seventh by hand, or to be able to multiply two two-digit numbers together by pencil and paper, or to be able

> to divide 10 by 1.05 without a calculator. Such drills are deadly dull, and the time must be spent instead to foster 'higher-order thinking skills'. I am not making up these examples; they were actually used in front of a roomful of witnesses this summer. I took notes. Those subject to bad teaching in years past (Just invert and multiply, kid, that's how we divide fractions, just *do* it) at least could *do* it. The kids today not only cannot do it, they cannot do anything *else* either. Except perhaps to invert matrices and find the "best-fit" line on a calculator.

My family came to USA soon after the "standards" was published. At first I knew very little about the so-called "reform" of the mathematical education in America, but I observed that my daughter' teacher of mathematics foisted a calculator into her hand all the time. Since we had no money to pay a private school, I understood that I had to do something radical and started to call my daughter "a victim of American education" whenever I saw a calculator in her hand. By the end of high school she was one of a few students who could calculate mentally. Many university freshmen grasped a calculator when they needed to calculate ten percent of a number, for example. Sometimes I tore calculator from a student's hand exclaiming: "You can do it without a calculator!" The student gazed at me for a while in astonishment, then realized that he really could, but I was the first person in his life who cared to tell him that it was worth while.

Around this time I was buying food in a grocery store, where eight oranges were sold for a dollar. I put (as I thought) eight oranges in a plastic bag and went to the cashier, who counted my oranges and said that there were only seven. I did not want to cross the hall for one orange and asked her to prorate the price. The young lady took out her calculator, but did not know what to calculate. I easily calculated it mentally, but kept silent to see what she would do. She called her supervisor, a young man with a big calculator, but he also could not figure it out. He counted the oranges again, found that there were eight and this settled the matter. So much about real world and high-level thinking skills.

The 1989 “standards” were critisized by many competent people including professional educators. Frank B. Allen, a former president of NCTM, declared:

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> sadly, the publication in 1989 of the first of the NCTM’s three ”Standards” reports (which are not standards because they do not set levels of student achievement) marked a drastic change in the Council’s status. Now, its hard-won reputation squandered by its shrill advocacy of failed procedures, the NCTM stands before the nation as a rogue organization whose Standards-based policies are largely responsible for the undeniable fact that school mathematics in the USA is a disaster.

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> THIS IS NOT INSTRUCTIVE MATHEMATICS. The standards-based subject (SBS) purveyed by the NCTM is so laden with major defects, so over-adjusted to alleged student learning deficiencies, that it no longer retains the properties of mathematics that make its study worthwhile. Mathematics is EXACT, ABSTRACT and LOGICALLY STRUCTURED. These are the ESSENTIAL and CHARACTERIZING properties of mathematics which enable it, WHEN PROPERLY TAUGHT to make unique and indispensable contributions to the education of all youth.

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> The philosophy of moral relativism, which condones deviate behavior and insists that nothing is really wrong, now dominates the mathematics classroom. Students must not be told that they are wrong because this might impair their ”self esteem” and the teacher might be seen as a judg???mental

> despot. Math must be made easy and fun. In earlier years it was well recognized that math, properly taught, is a difficult subject whose mastery requires hard work and sustained concentration. Education was seen as the process of ADJUSTING STUDENTS to the subject. Now, NCTM policy seeks to ADJUST THE SUBJECT to students and to whatever learning deficiences or "learning styles" they may have. THIS IS EDUCATION TURNED ON ITS HEAD. [**Allen**]

Now let us concentrate our attention on PSSM [**PSSM**]. Like its three-volume predecessor, PSSM is a strange and fancyful document. Like its predesessor, it has no index, and it is not structured accoring to mathemtical structure, there are no parts called "quadratic function" or "trigonometry" or "exponents and logarithms" or "combinatorics". Thus, if you want to know what it says about some particular mathematical topic, you have to browse all of it. Once I wanted to know what it says about geometrical theorems. First I looked into the five chapters called "Geometry": one overall and four pertaining to Pre-K-2, 3-5, 6-8 and 9-12 grades respectively. I found only one "theorem". It is in the 9-12 section (p. 314). More exactly, it is a "diagram that shows the use of coordinate geometry to prove that the medians of a triangle intersect" It is not even said that they intersect at one point. No argument is present, so it is not really a theorem. I found no mention of vectors; so much about connections. The book also has five chapters called "Reasoning and Proof" classified in the same way. I browsed them and found no proofs. After some more search I found a proof of the Pythagorean theorem in the section "Algebra" for the 9-12 grades (p. 301). I still don't know if there are any other theorems in the book.

This chaotic organization is an efficient defence. To criticize how PSSM treats some topic, you need to collect all that it writes about it, but this cannot be done without a careful study of the whole document. All potential critics are too busy for that, so

PSSM remains essentially uncriticised. It is interesting that PSSM aroused much less emotions of any kind that the 1989 "standards", although it is much less arrogant. Evident mistakes are corrected. The most pompous and irritating phraseology is eliminated. Calculators are not pushed so aggressively. The phrase "real-world problems" is completely absent. You may say that PSSM is a "hair-dressed" version of "standards", which nevertheless keeps its main feature: contempt for the structure of mathematics. Mathematical topics are not at home there: they appear without any order only when the authors have a fancy to invite them. Many important guests seem to be simply forgotten and nobody cared to check which. Perhaps, problems feel in that book better than theory? I looked for problems in the 9-12 section "Problem Solving" and found one:

Problem 52 How many rectangles are there on a standard 8×8 checkerboard? Count only those rectangles (including squares) whose sides lie on grid lines. For example, there are nine rectangles on a 2×2 board, as shown in figure 7.27. $\boxed{\textbf{P : rectangles}}$

In a Russian document such a problem would be put into a section "Combinatorics" and surrounded by congenial problems. However, in PSSM it stands alone like an exotic animal separated from its native land and put into a cage.

But let us return to our topic: word problems. I browsed all the book of PSSM and found only a few of them, all one-step. Perhaps, I missed something in chapters named "The Equity Principle", "The Teaching Principle", "The Learning Principle", "Communication", "Connections", or "Representation", but I have no time to spend it so unproductively.

It is remarkable how sterile is that noisy "reform" movement in American mathematical education: in more than twenty years of turmoil it produced **none** new good problem.

Of course, new good problems are invented in USA, for example by those who organize mathematical olympiads, but the Department of Education, NSF or NCTM never invite these people to make decisions.

Even organizations, which are openly for-profit, invent good problems sometimes. Look at this problem:

Let me finish this section with a large quote from a very interesting article written by a mother of Russian origin whose son attend a Fernch school in Maryland:

> I am often told that my child achieves good academic results because he is bright and would do well in any school. That is very nice to hear, but unfortunately, it is not true. My child does well when he is taught well. He has two teachers - his Russian teacher and the teacher at his French school - who both use time-honored, traditional methods of teaching. They do dictations, recitations, and repetitive rhythmic drills in grammar and spelling with their students. The methodology is specified in the scripted, sequential lesson plans that they both follow. The results are impressive.
>
> In his English classroom, on the other hand, where the teachers are not familiar with the notion of scripted or sequential curriculua, the results are quite different. The teachers improvise the program as they go along under the pretense of trying to suit it to individual class needs. My son had been doing nearly as poorly in these English classes as all of his classmates until I started tutoring him. After that, things quickly improved. It is true that my son is easy to teach, but you do have to teach him if you want him to learn. Left to his own devices, which is what the child-centered, unstructured instruction in his English classroom had essentially done, he invented spelling and sentence structure, without getting close to inventing the correct forms. His classmates, whose parents do not fill in the gaps left by the teachers, still invent spelling in fifth grade and some of them are still far from being fluent

readers. $< \cdots >$

One time when my son was eight and thoroughly confused by the homework his English teacher had given his class, he said: "Mom, why doesn't my French teacher teach my English teacher how to teach?" [**Kramer**]

23. Wars in American Mathematical Education

In October of 1999 the US Department of Education headed by Richard W. Riley approved ten K-12 mathematics programs by calling five of them "exemplary" and other five "promising". The approved programs are listed and described at [**programs**].

This decision was based on conclusions of an Expert Panel, most members of which have neved published a research article in mathematics. The list of members of the Expert Panel is available at [**panel**]. In fact, Manuel P. Berriozábal was the only member of the Panel with a substantial record of research in mathematics. Three years later he claimed:

> The panel consisted of 15 professionals mainly in the areas of mathematics and science education. To the best of my knowledge, I was the only mathematician with a prior documented record of traditional mathematics research.

$< \cdots >$

> I vainly advocated for a guideline that as a necessary condition for being designated as Promising or Exemplary, a reviewed candidate needed to demonstrate that students engaged in the candidate's program had been positively impacted by completing higher level college preparatory courses, by college attendance, bu college graduation and by majoring in mathematics-related areas. This proposal was rejected for two reasons: Most of the reviewed candidates would not be old enough to have compiled such data, and Congres-

sional legislation required that a list of Promising and Exemplary candidate designations be produced.

< · · · >

> In the years that we met, five candidates received an Exemplary designation and seven a Promising designation. I either voted against or abstained when these programs were considered because the impact review of each program in my opinion did not provide adequate evidence of success. [**Berr**]

Then Berriozábal quotes John Conway, head of the University of Tennesseeat Knoxville, say this:

> There is so much criticism of mathematics education in this country. In our calculus classes we all see unprepared students, and K-12 mathematics instruction and curriculum seems a ready focus for blame. In my view, I have met the enemy and he is us. Research mathematicians have for many years divorced themselves from what happens in K-12 mathematics. This means that we cede the entirety of the preparation of future college students to people with limited mathematics expertise and experience.

Conway is right. Although there are plenty of bright mathematicians in USA, for a long time they were not invited to participate in making important decisions about education. Sincerely speaking, they did not especially object. Like all people, mathematicians are prone to avoid extra work and sometimes say: "Why should I bother myself with public education? There are special people to care about it." So they did for a long time in America (but not in Russia). However, this time some of them decided to act. On November 18, 1999 The Washington Post published a letter signed by 200 mathematicians and other scientists, including some prominent educators, urging Riley to withdraw his department's approval. The letter is endorsed by seven Nobel laureates and winners of Fields Medal, the highest award in mathematics.

NCTM [**NCTM.letter**] immediately expressed a complete support for Riley's decision. This is understandable because the Expert Panel based its criteria at least partially on the "standards". However, it makes sense to look at that letter, so disgusting it is. Riley answered to the mathematicians' letter by reaffirming his position, see [**Riley**]

This is what David Klein writes about this situation:

> Although a clear majority of cosigners are mathematicians and scientists, it is sometimes overlooked that experiences education administrators at the state and national level, as well as educational psycho;ogists and education researchers, also endorsed the letter. $< \cdots >$
>
> The mathematics programs criticised by the open letter have common features. For example, they tend to overemphasize data analysis and statistics, which typically appear year after year, with redundant presentations. The far more important areas of arithmetic and algebra are radically de-emphasized. Many of the so-called higher-order thinking projects are just aimless activities, and genuine illumination of important mathematics ideas is rare. There is a near obscession with calculators, and basic skills are given short shrift and sometimes even disparaged. Overall, these curricula are watered-down math programs.

$< \cdots >$

> The U.S. Department of Education is not alone in endorsing watered-down, and even defective, math programs. The NCTM has also formally endorsed each of the U.S. Department of Education's model programs (`www.nctm.org/rileystatement.htm`), and the National Science Foundation (Education and Human Resources Division) funded several of them. How

> could such powerful organizations be so wrong?
>
> These organizations represent surprisingly narrow interests, and there is a revolving door between them. Expert panel member Steven Leinwand, whose personal connections with ”exemplary” curricula have already been noted, is also a member of the NCTM board of directors. Luther Williams, who as assistant director of the NSF approved the funding of several of the recommended curricula, also served on the expert panel that evaluated these same curricula. Jack Price, a member of the expert panel is a former president of NCTM, and Glenda Lappan, the association’s current president, is a coauthor of the ”exemplary” program CMP. [**Klein**]

Panel members’ conflicts of interests, were commented in the media. One article justly concluded:

> The review process seemed to deteriorate into a meeting of friends reviewing each other’s works and then using pseudo-scientific methods to bolster their claims. The process was inherently broken. [**KidsDoCount**]

H. Wu recently wrote:

> A second example is the announcement by the U.S. Department of Education in October of 1999 that ten mathematics programs were to be regarded as *Exemplary* or *Promising*. If a program can be considered among the ten best the nation has to offer, it may be taken for granted that the *mathematics* of each of these ten programs meets the minimum standard of being coherent and free of significant errors. Yet, to take the most obvious example, the mathematics of *Mathland*, one of the five *Promising* programs, can be objectively demonstrated to be shallow, incomplete, incoherent, and not infrequently just

> plain wrong. So how did this travesty come about? In a recent authoritative publication from the National Research Council *On Evaluating Curricular Effectiveness* ([Confrey-Stohl]), the inattention to mathematical content in the review process of the U.S. Department of Education is revealed. The Department appointed an Expert Panel to set up a procedure for examining the evidence of success of the submitted programs. According to Richard Askey of the University of Wisconsin, in the 48 reviews of the initial 12 exemplary or promising programs, “no mention of any mathematical errors was made” (see p. 79 of [Confrey-Stohl]).
>
> Now you must understand that any of the existing curricula, old or new, is so riddled with errors that it would take a superhuman mental effort to blot them out. How then did the dozen or so Expert Panel members and almost 95 Quality Control Panel members manage *not* to notice any of these glaring errors? One reason may be because there was only one mathematician on the Expert Panel, and only two on the Quality Control Panel. [**Wu.content**], p. 8

Thus we observe an open confrontation between mathematicians and scientists on one side and educational officials and leaders on the other. Is it only about errors and conflict of interests?

Not only. Another evident point of confrontation is whether children should be taught paper-and-pencil arithmetical algorithms or use calculators instead of that. The difference of opinions can be illustrated by two quotes, both included into the mathematicians' letter. One is from an article written by Steven Leinwand, the co-chair of the Expert Panel (who also was at different years one of directors of NCTM and member of advisory boards for three programs that were being evaluated by the panel), entitled "It's Time To Abandon Computational Algorithms" and published on February

9, 1994, in Education Week on the Web [**Leinwand**]:

> It's time to recognize that, for many students, real mathematical power, on the one hand, and facility with multidigit, pencil-and-paper computational algorithms, on the other, are mutually exclusive. In fact, it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous.

The other quote is from a report made by a committee formed by AMS for the purpose of representing its views to NCTM:

> We would like to emphasize that the standard algorithms of arithmetic are more than just 'ways to get the answer' – that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials. [**AMS.report**]

Pay attention that this statement was made only in 1997 and published only in 1998, which was too late because the "standards" had recommended the usage of calculators instead of paper and pencil already in 1989 and at that time AMS seemed to support it.

In Russia mental and pencil-and-paper computations were always recommended throughout the school and considered essential for understanding. For example, Igor Arnold wrote in his book "Logarithms in the school course of algebra" (I don't have this book with me and quote from memory): "We tell students that $\log_{10} 2 \approx 0.30103$ because one can obtains 2 by raising 10 to this degree, but the problem is that the student has never seen anybody 'obtain' 2 in this way." In view of this, Arnold recommended to teach students to estimate logarithms by mental calculations and by hand, without tables. (Calculators were not available in the 30s, when Arnold wrote his book, but it is clear that he would say the same if they were.) For example, $2^{10} = 1024$, which is a little more than 10^3, whence $\log_{10} 2$ is a little more than 0.3. Hence $\log_{10} 5$ is a little less than 0.7, $\log_{10} 4$ is a little more than 0.6 and $\log_{10} 8$ is a little more than 0.9. After that, using interpolation, we can estimate that $\log_{10} 9$ is a little more than 0.95, whence $\log_{10} 3 \approx 0.48$, whence $\log_{10} 6 \approx 0.78$. Interpolation between 6 and 8 gives $\log_{10} 7 \approx 0.84$. There are many other numerical relations which allow to check and improve these estimations. I believe that mental and by hand estimations are very useful in all areas of mathematics, including trigonometry and study of functions in general, and that they are essential as a preparation for calculus.

Thus "reformers" of mathematical education in USA have already excluded most theory and now want to exclude arithmetical algorithms from curriculum. What for? For the sake of problem solving. American educators have said many times that they care very much about problem solving. "Agenda for Action", which expressed the NCTM's vision twenty years ago, suggested that "problem solving be the focus of school mathematics $< \cdots >$" [**Agenda**], p. 1. The "standards" completely share this opinion (p. 6) and start every part with a chapter called "Mathematics as problem solving". Every section of PSSM also contains a chapter named "Problem Solving". What is ridiculous is that this chapter is there along with others, including "Algebra" and "Geometry" for example. So, according to the authors of PSSM, there is problem solving without any partilular topic and there are algebra and geometry without problem solving.

In my opinion, solving problems is really very important, I even suggest that public mathematical education **IS** teaching children to solve mathematical problems, provided the words "problem" and "to solve" are interpreted properly. After all, any mathematical theory can be represented as a series of problems. So my first reaction to these declarations was positive. Suspicion came to me only when I noticed that every time when these educators declare their concern for problem solving, they try to exclude something from curriculum. If all their proposals are accepted, nothing remains but mirages.

Meanwhile the educational war continued. On February 2, 2000 there was a hearing on "The Federal Role in K-12 Mathematics Reform" [**hearing**], where opposite opinions were presented in a very sharp manner. In particular, Jim Milgram, a mathematician, mentioned "dramatic drop in content knowledge that we have been seeing in the students coming to the universities in recent years". I believe that Milgram is right. If you live on the top of a building, you have to care about the whole building.

I am not satisfied with American mathematicians' stand. They defended arithmetical operations and this was very good of them. But is this enough? Certainly, not. The mathematicians should go further and insist on content beyond arithmetical operations. Till now they have not done it.

24. Contempt for The Structure of Mathematics

We often say that American schools do not present advanced enough material. This is essentially true, but sometimes disguised by unproportional ambitions.

What about the "standards", its idea of advanced topics can be illustrated by the following quotes:

> Prior to the work of the ancient Greeks (e.g. Thales and Pythagoras), geometricideas were tied directly to the solution of real-world problems. Hence, the subsequent abstraction and formalization of these ideas, which evolved into the subject of geometry as we know it, has always had many applications in the real world. More recently, fractal geometry, which originated in

> the mid-1970s with the pioneering work of Benoit Mandelbrot, has provided useful models for analyzing a wide variety of phenomena, froom changes in coastlines to chaotic fluctuations in commodity prices. It is the intent of this standard that, whenever possible, real-world situations will provide a context for both introducing and applying geometric topics. (p. 157)

This excurs into the fractal geometry is, of course, quite superficial. (The very word "dimension" is not mentioned at all.) The idea to teach fractals in school has already found many supporters. (Everything is possible for those who are not competent enough to understand how difficult it is.) I asked several school teachers who were enthusiastic about teaching fractals to define a fractal and none of them mentioned the idea of dimension, least defined it. Usually they emphasized "repeating patterns". When I asked why they were not satisfied with wall-paper, they took offence.

Why fractals? I think, it is because for many years American educators have been accused of "dumbing down" their students and now they desperately try to show that they care about some advanced topics, but some of them are not competent enough to choose these advanced topics realistically.

Katherine K. Merseth, a director of an educational center, having mentioned really interesting experimental results, writes:

$< \cdots >$

> many individuals believe that mathematics is a static body of knowledge. And what is taught in school reinforces this notion: most schools currently teach eight years of 18^{th}-century "shopkeeper" arithmetics, followed by a year of 17^{th}-century algebra and a year of geometry, basically developed in the third century B.C. Even calculus, as taught in today's schools, is three centuries old. Very few students and adults are aware that, with advances in such

> areas as fractals, discrete mathematics, and knot theory, more mathematics has been discovered in the last 35 years than in all previous history. Little of this new material, however, makes it into the schools or the public discourse. [**Merseth**]

Similar vain ambitions exist in other countries with a low quality of mathematical education. For example, Suely Druck, the President of the Brazilian Mathematical Society, lately had to declare:

> ... only 5.99% of high school students reach the desired level and at the 4-th grade of elementary school only 6.78% $< \cdots >$only 21% of the population can understand information presented in graphs and tables $< \cdots >$Ignorance leads to even more disastrous results: in the name of modernization of mathematical education, in some schools the Pythagorean theorem was excluded as "obsolete". [**Druck**]

Another example: on the same p. 157 the "standards" recommend to "develop an understanding of an axiomatic system through investigating and comparing various geometries", which is widely interpreted as a suggestion to teach non-Euclidean geometry, which is impossible to do having eliminated almost all the logical structure.

Still another example:

> College-intending students should become familiar with such distributions as the normal, Student's t, Poisson, and chi square. Students should be able to determine when it is appropriate to use these distributions in statistical analysis (e,g., to obtain confidence intervals or to test hypotheses). Instructional activities should focus on the logic behind the process in addition to the "test" itself. [**St.1**], p. 169.

Do the authors know that the theory of continuous random variables, where normal, Student's and chi square distributions belong, needs such an amount of calculus as a prerequisite, which usually takes three semesters? Do they know that calculus also needs certain prerequisites, which take years to teach, but are too often neglected in America? (According to TIMSS, among all students in the world who take calculus in school, American students are very low in knowledge of pre-calculus topics.) The authors of "standards" want to reform American mathematical education, but actually only aggravate its main shortcoming: vain ambitions and contempt for consistent, systematic and thorough study. Absence of any organized curriculum made neglect for prerequisites a chronic disease of American education. Some educators even have the arrogance to declare it as a principle. For example, the Hawaii state standards got F at the Fordham scale and with a reason. They claim:

> Learning higher-level mathematics concepts and processes are not necessarily dependent upon "prerequisite" knowledge and skills. The traditional notion that students cannot learn concepts from Algebra and above (higher-level course content) if they don't have the basic skill operations of addition, subtraction, etc. has been contradicted by evidence to the contrary. [**Fordham.Klein**], p. 56

Also, what do the authors of "Standards"mean by "becoming familiar"? It has to be something quite superficial. Nowadays all American children are familiar with the ideas of cosmic travel, travel in time and robots because such films as "Star Trek", "Terminator" and "Robocop" brought these ideas into every home. This is not bad at all, but this does not make children capable to operate space ships or design robots. It is necessary to distinguish the useful, but superficial level of familiarity, provided by the entertainment industry, from the mastery and understanding to be achieved in school. Are the authors of "standards" aware of this difference?

25. State Standards and Ignorance of American Educators

The Fordham Foundation has earned a name in Mathematical Education by publishing sharp but well-founded criticisms of state standards of mathematical education. The last issue, written by David Klein and several collaborators, appeared in January 2005 [**Fordham.Klein**].

According to this report, only three states, namely California, Indiana and Massachusetts received A, that is “excellent”. Three states, namely Alabama, New Mexico and Georgia received B. Other 45 states received: C – 15 states, D – 19 states, and F – 11 states. Pehaps, the report is too harsh and the standards are not so bad? No, its concrete criticisms and quotes from state standards convince me that the grades are fair, perhaps even liberal.

These dreadful standards show in which isolation have many American educrats put themselves. In the world greatest super-power with hundreds of universities the state bosses in mathematical education found nobody to help them write something more decent. No wonder that the report claims:

> Mathematical ignorance among standards writers is the greatest impediment to improvement. (p. 24)

On the next page the report proposes “Four Antidotes to Faulty State Standards, the first of which recommends:

> Replace the authors of low-quality standards documents with people who thoroughly understand the subject of mathematics, include university professors from mathematics departments.

But let us concentrate on our topic: word problems. On p. 11 in a section “Mathematical Reasoning and Problem Solving" the Fordham report writes:

> Problem-solving is an indispensable part of learning mathematics and, ideally, straightforward practice problems should gradually give way to more difficult problems as students master more skills. Children should solve single-step word problems in the earliest grades and deal with increasingly more challenging, multi-step problems as they progress. Unfortunately, few states offer standards that guide the development of problem-solving in a useful way. Likewise, mathematical reasoning should be an integral part of the content at all grade levels. Too many states fail to develop important prerequisites before introducing advanced topics such as calculus. This degrades mathematics standards into what might be termed "math appreciation".

This is very well said. I would be very happy if this wish were supported by reality. But I have doubts. Can Klein and his colleagues guide me to web sites where the vaunted California, Indiana and Massachusetts put samples of problems which they like? Till now I have seen very little of this.

This is another telling example of this tendency. On pp. 17-18 the Fordham report [**Fordham.Klein**] writes:

> The attention given to patterns in state standards verges on the obsessive. In a typical state document, students are asked, through a broad span of grade-levels, to create, identify, examine, describe, extend, and find "the rule" for repeating, growing, and shrinking patterns, as well as where the patterns may be found in numbers, shapes, tables, and graphs. $< \cdots >$ The following South Dakota fourth-grade standard is an example of false doctrine (a notion explained in greater detail on page 34) that is representative of standards in many other state documents.

Students are able to solve problems involving pattern identification and completion of patterns. Example: What are the next two numbers in the sequence? Sequence...

The sequence "1, 3, 7, 13, __, __" is then given. The presumption here is that there is a unique correct answer for the next two terms of the sequence, and by implication, for other number number sequences, such as: 2, 4, 6, __, __, and so forth. How should the blanks be filled for this example? The pattern might be continued in this way: 2, 4, 6, 8, 10, etc. But it might also be continued this way: 2, 4, 6, 2, 4, 6, 2, 4, 6. Other continuations include: 2, 4, 6, 4, 2, 4, 6, 4, 2 or 2, 4, 6, 5, 2, 4, 6, 5. $< \cdots >$ Given only the first four terms of a pattern, there are infinitely many systematic, and even polynomial, ways to continue the pattern, and there are *no possible incorrect* fifth and sixth terms. Advocating otherwise is both false and confusing to students.

26. Word Problems as A Skapegoat

Since word problems are difficult for some teachers, it would be natural to say: "It is regretful that we are weak in solving word problems. We should pay more attention to them in schools of education. Textbooks and exams should contain more varied word problems. Educational journals should publish articles instructing teachers how to teach word problems. Textbooks should place word problems in a reasonable order, starting from easy ones and gradually increasing their difficulty and reaching quite difficult ones at the end." All this was done in Russia to some extent for a long time and is done now, but it seems that nothing of this was done for a long time in USA. Instead, educational leaders tried to create impression that there was something wrong with word problems themselves. We have already seen how "standards" blamed word problems "by type" for the teachers' incompetence.

Here is another example: "Mathematics Teacher" (the main American journal for high school teachers of mathematics) published an article written by Zalman Usiskin, an influential educator, where he recommended to delete the "traditional word problems" from curriculum. He wrote:

> The traditional word problems (coin, age, mixture, distance-rate-time, and digit) are in the curriculum because of a very valuable goal, the goal of translating from the real world into mathematics. But except for mixture problems, they do not help achieve that goal. [**Usiskin.not**], p. 158-159

On the same page he wrote: "Algebra has so many real applications that traditional phony word problems are not needed."

Why does Usiskin call traditional word problems phony? He quotes the problem "A person has 20 coins in his pocket, some nickels and some dimes. The total value is $1.75. How many nickels and how many dimes were there?" and continues: "Since the coins were counted, shouldn't the counter have kept track of the number of dimes?" (p. 159).

In Russia (and in most countries, as I believe) this strange argument would be ignored as a bad joke, only in America it is taken seriously. This argument was originally proposed by Edward L. Thorndike, a well-known American behaviorist. Although the book [**Thorndike**] names several authors, Thorndike was the leader, so I refer to him as the author. One chapter of this book, called "Unreal and useless problems", applies this label to all those problems which cannot be faced in real life literally. Thorndike thought that such problems produce a sense of futility and proposed to exclude all of them from curriculum. This sense of futility is complete nonsense and more attentive teachers noticed it. For example, Breslich wrote:

> If puzzle problems are presented in the true light, the objections to a limited

> number of such problems in a course in algebra will cease. They may even prove to be more interesting, stimulating, and profitable than some of the problems that are considered to be real. $< \cdots >$
>
> Powell reports that in the majority of cases they [pupils] show little concern about the problems which the teachers rated as highly interesting. Pupils are not greatly worried about the reality of the problems they have to solve. [**Breslich**], p. 185

My observations also contradict Thorndike's claim. For example, the problems 21 $\boxed{P: geese}$, 8 $\boxed{P: witches}$, 9 $\boxed{P: Ivan}$ mentioned above are evidently fantastic, but they are liked by Russian children. I have no doubt that American children, like all children in the world, have fantasy and well may be interested in problems which Thorndike classifies as "unreal and useless". We have been told repeatedly that American students of mathematics lack mjotivation. May be the obscession of American educators with "real" problems is one cause of it?

Since Usiskin declared that "Algebra has so many real applications", we may expect enough of them in his program UCSMP for Grades 7-12 declared "promising" by the Expert Panel, but this is not the case. The "Mathematically Correct Algebra 1 Reviews" [**MathematicallyCorrect**] rated this program the lowest in "quality and sufficiency of student work" and said about it:

$< \cdots >$

> there are far too few problems for each subtopic and they fail to cover the upper difficulty levels. The coverage of word problems is especially weak as there is no good introduction to writing equations with variables for unknowns, far too little practice on this, and no word problems beyond the easy level.

Where are those "so many real applications"?

Also we may expect to find "real applications" of algebra in the "Sampling of Algebra Problems" included in his article with a promising title "Why Is Algebra Important To Learn" [**Usiskin.why**], p. 34. This publication gave the prominent educator an excellent opportunity to illistrate his theses. But look at the first problem in the sampling:

Problem 53 The World Almanac and Book of Facts 1995 lists 59 major earthquakes from 1940 to 1994. Here are their frequencies by season of the year: Autumn, 14; Winter, 14; Spring, 11; Summer, 20. Use statistics to determinate whether these frequencies support a view that more earthquakes occur at certain times of the year than at others, or if differences like these occur commonly by mere chance. (quadratic expressions)

Usiskin's own solution?

This is an exercise in application of chi-square criterion, which well may be used in a university course of mathematical statistics. In such a course a teacher would discuss conditions under which this criterion can be applied, which Usiskin does not do because he pretends that this problem can be solved at high school. To take such a course, the students need already to be acquainted with a theory of continuous random variables, which in its turn is based on several semesters of calculus. What Usiskin did show is that university courses of mathematics and statistics are important to learn to become a professional statistician. But we knew it very well without him. This does not yet mean that *school algebra* is important to learn for those who will not become a professional scientist. Now let us look at the second problem in the sampling:

Problem 54 To estimate the number N of bricks needed in a wall, some bricklayers use the formula N=7LH, where L and H are the length and height of the wall in feet. About how many bricks would a bricklayer need for a wall 8.5 feet high and 24.5 feet long? (formulas)

If Usiskin wanted to give students just an exercise of plugging numbers into a formula, here it is. However, my vision of school mathematics is different from his. I want children to *understand* the world around them. Does this formula depend on thickness of the wall and parameters of a standard brick? If does, how? Is cement layer between bricks also important? Why L and H are multiplied in the given formula? Why aren't they added together? (Compare Kenschaft's observation in the section 12 *Ignorance* of this article.) Usiskin does not suggest to discuss all this. He makes no hint that the teacher might involve students in *deducing* this formula from parameters of real bricks (which are easy to find at any construction site), usual thickness of walls and cement layer and properties of volumes – which might be a fruitful activity. Also there is something strange with dimension here: if the number 7 has no dimension, then the formula 7LH has dimension of square feet. How can it be equal to N which has no dimension? Usiskin seems not to care about dimension, but I certainly do.

If we look at the other problems in Usiskin's sampling, we find that all of them are stretched to make impression of relevance to various aspects of life, but every time this relevance is far-fetched. (I have discussed one of these problems in [**Toom.Child**].) Thus Usiskin's thesis that school algebra has many real applications remains unproved. Meanwhie (I think that it will be a very long meanwhile), he should not discourage teachers from using traditional word prioblems.

Let me emphasize that I don't imply that mathematics has no applications. I am not so silly. What I mean is that most word problems used in education at various levels are not applications and pretending that they are only misleads and disappoints the students. For example, if an algebra problem speaks about pipes bringing water into a pool, it is *not* an application of algebra to pool management. It is algebra. If a problem asks in how many ways can a committee of ten elect a chair, a secretary and a treasurer, it is *not* an application of combinatorics to committee work. It is combinatorics. If educators cannot yet explain this important issue in professional terms, they at least should not confuse it with pejorative terms.

On the other hand, there are good problems in applied mathematics and there are valuable collections of them, but all of them need much more than American school mathematics. Let us take one of these collections [**Klamkin**] and list the titles of its sections:

> Mechanics, electrical resistance, probability, combinatorics, series, special functions, ordinary differential equations, partial differential equations, definite integrals, integral equations, matrices and determinants, numerical approximations and asymptotic expansions, inequalities, optimization, graph theory, geometry, polynomials, simultaneous equations, identities, zeros, functional equations, miscellaneous.

To solve these problems, even to understand what most of them ask, one needs to know quite substantial mathematics. But we are speaking about a much more initial stage of learning mathematics. Solving word problems at this stage is not a preparation for professional activity in the corresponding area. For example, if a problem involves cars moving towards each other, it is **not** a preparation for future managers of street traffic. Rather these cars become in the minds of students their mental models for dealing with abstractions: variables and relations between them. Every mathematician knows how important is imagination in doing mathematics. To deal with abstractions we need to represent them mentally in various ways. **Solving word problems is a mean to develop imagination to deal with scientific abstractions.**

27. Really-Really Real... Really?

Already for several decades American educators are desperately looking for really-really real problems and fail, one genertion after another.

Yes, there are bombastic promises to create really-real 'real-world' problems, but what is their quality ? What we see in quantities are the same old problems, only encumbred with irrelevant details. For example, if a geometrical problem is about a cylindric can of coffee, the texbooks tend to inform the reader which firm produced this brand, what is its price and give a photo of this can in the margin. All this is to give a superficial impression of relevance to real life. And coffee industry may be listed among 'real-life' applications covered by this textbook! But students are usually irritated by irrelevant data in problems, and they are right. Life is too short to waste it sorting out irrelevant information. Thus the 'real-life' bandwagon splits problems into two disjoint extremes. On one side are problems formulated in purely mathematical terms like 'factorize a given polynomial' or 'solve a given equation'. On the other side are cumbersome story problems pretending to be really-really real, but in fact only boring.

In the beginning of this paper I wrote that we cannot see, hear, touch or smell abstractions. In the literal sense this is true. But we could not do mathematics if we found no way to bypass this. I have watched students explaining something to each other. I think that this is the most valuable experience they had in my classes of College Algebra. When they made gestures imitating such 'realities' as moving cars or current in a river, they made abstractions almost visible and touchable. I say 'abstractions' because these cars and current are not real and this is their great advantage. Since cars, ships, pumps and other 'realities', mentioned in word problems, are devoid of irrelevant details, they can serve as semi-abstractions, still understandable for novices. This makes word problems an excellent breeding-ground for initial study of mathematics and science. After discussions, my students write equations, where every sign has roots in their visual and motor experience. The feeling of understanding which they experience in this way is the most appropriate reward for doing mathematics. This reward actually coincides with the purpose and result of teaching.

Remember the relation between “forward” and “inverse” problems which we mentioned. It reminds of the idea of “reversibility” which plays an important role in Piaget’s theory. Indeed, we cannot call a study successful if a student can solve problems in only one direction. But it is impossible for all these directions to be equally relevant to everyday needs ! Remember how much effort do physicists apply to deduce observable phenomena from postulates of their theories. Should they obey educators who say that it is useless ?

It is not new that science and enlightment ask questions which go beyond small-minded gains. What is new is the anti-intellectual position of some educators. Those educators who require immediate ‘real-world’ relevance, pretend that they care about students, but real students have quite different concerns. In my classes it is me who reminds students that we make certain assumptions when we solve word problems. For example, if Ann and Mary do a certain job together, I may say: “We assume that Ann and Mary always work with a constant rate, like machines. In real life they may chatter about fashions.” The students laugh and take such comments as mere jokes. Their main problem in these formative years is quite different: it is personal development and self-organization. The assumption that Ann and Mary have a constant rate is real enough for them at this moment; the problem is how to relate such an assumption, expressed in words, to that sophisticated game with symbols which is called algebra.

Generally speaking, all human cognition simplifies reality. And it would be very silly to say that the closer to reality the better. This would mean that a police report is more valuable than a poem or that a photo of a mountain is more valuable than Hokusai’s drawing or that a pile of experimental data is more valuable than the ‘simple’ equation concluded from or checked by these experiments.

We are dealing here with some of the most fundamental laws of culture: human culture never describes reality one-to-one. Instead, it always condences, simplifies, idealizes. The simplest example: geographical maps. Are they equal to the landscapes they represent ? Instead of being identical with reality, creations of human mind are subject to their own laws. One of these laws is the law of economy: any redundancy should be avoided, every detail must serve the purpose. The famous Russian playwright Anton Chekhov said that if a gun is hanging on a wall in the first act of a play, it must shoot in the last act.

A good problem shares all the same attributes which Bentley mentioned in connection with one of his excellent computer programs: "General Chuck Yeager (the first person to fly faster than sound) praised an airplane's engine system with the words "simple, few parts, easy to maintain, very strong"; this program shares those attributes." [**Bentley**], p. 6. Many traditional word problems share these attributes. There is plenty of waste, redundancy, confusion and boredom in the real world, all of which should be excluded from classroom.

Part V. Theoretical Background

28. Cognitive Development. Its Cultural and Social Foundations

Nowadays American educational literature is full of appeals to make mathematical education as close to reality and everyday life as possible.

What do we know about cognitive development of people who have always solved only practical problems with real data? This question is answered by several expeditions into regions populated by people belonging to so-called 'traditional' cultures. The scientists observed that these people do not solve even simple word problems if these problems go beyond their experience, although they can perform arithmetical operations. This is what some scientists wrote:

Luria about his expedition to Central Asia [**Luria**], p. 120:

> Subjects who lived in remote villages and had not been influenced by school instruction were incapable of solving even the simplest problems. The reason did not involve difficulties in direct computation (the subjects handled these fairly easily, using special procedures to make them more specific). The basic difficulty lay in abstracting the conditions of the problem from extraneous practical experience, in reasoning within the limits of a closed logical system, and in deriving the appropriate answer from a system of reasoning determined by the logic of the problem rather than graphic practical experience.

Cole & Scribner about their expedition to Africa [**CS**], p. 162:

> Experimenter: Spider and black deer always eat together. Spider is eating. Is black deer eating?
>
> Subject: But I was not there. How can I answer such a question?

Scribner [**Scribner**], p. 155:

> Both Luria and Cole identified this empirical bias as an important determinant of the poor problem performance of nonliterate traditional people..."

The most interesting observation made by these scientists is that "traditional" (that is, belonging to trditional cultures) subjects to not solve simple syllogistic problems, that is problems to solve which it is necessary and suficient to perform one syllogism. This does not mean that they try to solve these problems in our sense and fail or make mistakes. This means that they refuse to make statements which are not substantiated by their personal experience. One of Luria's subjects said exactly this: "We speak about what we have seen. What we have not seen, we do not speak about."

Based on his and his predecessors' observations, Tulviste in his book [**Tulviste**] came to certain conclusions, which I take the liberty to resume as follows [**Toom.Ind**]:

> Modern man has at least two qualitatively different modes of thinking. **The empirical mode** deals with directly observable material things and facts known from personal experience. Words in this case are used to denote real things or classes of similar things. People used and continue to use this mode when it is appropriate, for example in everyday life.
>
> **The scientific mode** deals with scientific notions rather than with concrete things. Meanings of words in this case cannot be explained just by pointing at objects. These meanings form a system an it is necessary to learn systematicallty to master it. The scientific mode appeared in a relatively recent hostorical time, probably in the Ancient Greece. Since that time it is transmitted from one generation to another mostly through schooling.
>
> Every "modern" man (we use inverted commas because those subjects who refused to solve simple syllogistic problems also are our contemporaries) can use the scientific mode because he studied at school and uses it when finds it appropriate, for example when solving the syllogistic problems, although solving them as such in not practicized at school.

Based on his theory, Tulviste wrote in his book:

> Knowing how to solve 'school' problems is, of course, not an end in itself. In school, pupils are taught primarily scientific information and scientific thinking. It would be impossible to create, confirm and use scientific information if

> every separate deduction had to be compared each time with reality or with available information on reality [**Tulviste**], p. 122.

If this theory is correct (I think that essentially it is), we have an explanation of the futility of one century of attempts of Americn educators to connect school with everyday life. The mission of school is to develop scientific thinking, whose main area of application is not trifles of everyday life. Knowing how to prove theorems *will not* provide you a bigger piece of pizza tomorrow.

Let me quote another observation made by Tulviste [**Tulviste**], p. 131:

> In a classroom in the Volochanka school where we did a study, above the blackboard hung a square, a circle, and a triangle cut from cardboard. The teacher told us that when she was a pupil, she found it very difficult to understand what geometric figures represent, what kind of objects or things they were. She did not know such 'things' at home. Now, having hung them above the blackboard, from time to time she repeated their names and descriptions for the pupils, beginning with the first grade, so that they would get used to the existence of 'things' of this type, ideal objects that can be described only within a specific system of concepts, and not by reference to immediate reality.

So that Siberian teacher tries to make her students get acqainted with abstractions, beginning with the first grade. At the same time some American educators are so 'advanced' that they try to deprive students from 'things' which have no reference to immediate reality.

I assume that my readers believe (as I do) that there are no inferior and superior races. There are two important conclusions from this belief. One is well-known: that children of all races can learn well and become competent members of a civilized society if properly educated. The other is equally true but often overlooked: that children of all races can learn poorly and fail to master some skills which are necessary for participation in a modern society if properly miseducated. Let us play the devil's advocate and think, how should one miseducate modern children to keep them on that archaic level of cognition which was described in [**CS, Luria, Scribner, Tulviste**] ? It seems very promising in this respect to disseminate in schools that *empirical bias* about which Scribner wrote (see above) and to remove from sight all problems which have no straightforward practical application. But this is very close to what some educators actually are trying to do ! All anthropologists cited above consider it as a drastic limitation that people of 'traditional' cultures cannot mentally accept an assumption like "spider and black deer always eat together" and make logical conclusions from it. But some modern educators try *intentionally* to remove from the curriculum problems where the students need to mentally accept a certain assumption like "John and Mary always peel potatoes with one and the same rate" and make logical conclusions from it. Actually these educators are trying to deschool the school and to demathematize the mathematics by demanding literal 'real-world' relevance.

Of course, I am not the first person to speak about connection between observations mentioned in this section and problems of modern man's cognitive psychology. L. S. Vygotsky did it long before me. In his letters sent to A. R. Luria during his expedition to Central Asia, L. S. Vygotsky wrote:

> Dear Alexander Romanovich. I am writing this literally in emphasia – in some elan, which one seldom can experience. I got Report $N^o 3$, protocols of experiments. I don't remember a more light and enjoyable day lately. This is literally like locks of several psychological problems opened with a key. This is my impression. The first rate importance of [your] experiments is beyond

> doubt for me, our new way is conquered now (by you) not only in idea, but in deed – in experiment. [**Luria.Helen**], p. 55

< · · · >

> I continue to think and shall think now, until somebody unconvince me, that it has been proved experimentally (on factual material more rich than in any ethnopsychological study, and more clean and correct than Levy-Brühl's) the philogenetic existence of a layer of complex thinking and a new structure, which depends on it, of all the main systems of psyche, all the most important kinds of activity – and in perspective – of the conscience itself. [**Luria.Helen**], p. 56

The chapter, from which I quoted, is entitled *"Fergana, dear Fergana..." expeditions to the Central Asia; demolition of Vygotsky's school.* Vygotsky's theory was one of those demolished by Soviet authorities. Only after Stalin's death Vygotsky's books became available again and his theory returned to university curricula.

29. Mental Discipline

To understand better the origin of those far-fetched ideas, widespread in Americn education, resulting in expulsion of abstract thinking from curriculum, we need to go to the beginning of the 20-th century when Edward L. Thorndike and several other thinkers created an enormous confusion in American education.

Throughout recorded human history it was a common opinion, even a commonplace that study of mathematics promotes general intellectual development. This is just one expression of this idea: a quote from Abraham Lincoln's "Short Autobiography" (Lincoln calls himself in the third person):

> He studied and nearly mastered the six books of Euclid (geometry) since he

> was a member of congress. He began a course of rigid mental discipline with the intent to improve his facilities, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow lawyers, half a dozen in a room, filled the air with interminable snoring.

The idea, expressed by Lincoln, was shared by many others, practically by all intellectuals for centuries. However, in USA in the beginning of the XX-th century the opposite idea emerged and remained dominant for decades. Due to excellent studies by Ravitch [**Ravitch**] and Kolesnik [**Koles**] we have a rich sample of quotes. The disaster in the psychological foundations of mathematical education was described by contemporaries in these terms:

> Paul Shorey: I have taken for granted the general belief of educators, statesmen, and the man in the street, from Plato and Aristotle to John Stuart Mill, Faraday, Lincoln, President Taft, and Anatole France, that there is such a thing as intellectual discipline, and that some studies are a better mental gymnastics than others. This, like other notions of "common sense," is subject to all due qualifications and limitations. But it is now denied altogether, and the authority of Plato, Mill, Faradey, or Lincoln is met by the names of Hinsdale, O'Shea, Bagley, Horn, Thorndike, Bolton and De Garmo. Tastes in authority differ. But these authorities are cited, not as authorities, but as experts who have proved by scientific method and ratiocination that mental discipline is a myth. There is no such proof, and no prospect of it. [**Koles**], p. 55

> Nicholas Murray Butler: As a result of a few hopelessly superficial and

> irrelevant experiments, it was one day announced from various psychological laboratories that there was no such thing as general discipline and general capacity, but that all disciplines were particular and that all capacities were specific. The arrant nonsense of this and the flat contradiction given to it by human observation and human experience went for nothing, and this new notion spread abroad among the homes and schools of the United States to the undoing of the effectiveness of our American education. [**Koles**], p. 56

Kolesnik quotes several other statements in a similar vein. In a nutshell, around the beginning of XX century American educators started to demand a direct utilitarian use of every piece of every school subject. If such use could not be presented, the "useless" topic was to be excluded.

For example, on p. 137 of [**Thorndike**] Thorndike starts a section *Genuineness* in which he writes:

> Relatively few of the problems now in use are genuine. First of all, over half of them are problems where in the ordinary course of events the data given to secure the answer would themselves be secured from the knowledge of the answer. For example, "In ten years John will be half as old as his father. In twenty years he will be three-fifth as old as his father. How old is John now? How old is his father?" In reality such a problem would only occur in the remore contingency that someone knowing that John was 10 and his father 30, figured out these future age ratios, then forgot the original 10 and 30, but remembered what the future ratios were!

Thorndike called 'genuine' only those problems which had immediate and literal practical relevance and proposed to minimize presence of all the other problems in the school curriculum. Our paper is only about word problems, but just one question: What remains of Euclid's 'Elements' after this 'real-world' purge?

When I came to America and became aware of such arguments, I first ignored them as absurdist jokes, so crazy they seemed to me. It took me several years or communication and reading educational literature to accustom myself to the strange fact that arguments of this sort are taken seriously, even respectfully and really influence curriculum and manner of teaching.

Let me make a comparison. Imagine that prospective teachers of literature in a certain country are made believe through their professional preparation that all fairy tales, fables, fantastic stories are useless. When told a fable, where animals speak to each other, they cannot comprehend and enjoy it in a normal way, as all children do, but exhaust their imagination in figuring out how could it happen in real life: perhaps, animals were especially trained to speak? perhaps, they were made some operation? perhaps, it were disguised people? etc. This is similar to the approach of some American educators towards word problems: they insist that it should be possible for the situation and for the question asked to take place in reality. Actually these educators suffer from some sort of mental deficiency which is not innate but artificially created by their professional preparation. Take Aesop's fable "The Crow and the Fox". From that strange viewpoint, which Thorndike successfully disseminated among American educators, this fable is useful only for those who have a chance in some future to perch on a tree branch with a piece of cheese in their mouths.

What made critics of this practical movement especially sad was not just that several scholars made a wrong decision, but a complete absence of critical thinking and common sense among them. How could all this happen? On p. 137 of his book Kolesnik quotes one of the actors of this drama:

> As William C. Bagley points out, Thorndike and Woodworth's investigations were published at a time when large groups of unselected students were beginning to swell the high school population. "It was inevitable," he says, "that any theory which justified or rationalized the loosening of standards should be received with favor." $< \cdots >$ Since mental discipline "stood squarely in the way of the movement that was opening the high school to the masses," Bagley expands, "anything that would tend to discredit this doctrine was seized upon with avidity."

Let me remind the reader that nowadays **all** children of school age are supposed to be at school, not only in USA, but in most countries including Brazil and Russia. It is true that such expansions inevitably bring with them some decline of quality. The best thing to do is
a) to admit with *regret* that this decline is inevitable for that moment of time,
b) to protect the more capable and better prepared students by creating special advanced programs for them and
c) to use every opportunity to upgrade the general program.
Regretfully, influential American educators reacted in other ways: they **embraced** the decline, called it *progressivism*, demagogically associated it with democracy and created appearance of "scientific" base for it. All this is not so much my own idea, but rather my conclusions from reading Diane Ravitch's excellent book [**Ravitch**].

Diane Ravitch wrote [**Ravitch**], p. 69:

> Despite his critics, however, Thorndike's views continued to have enormous currency; he was, after all, a towering figure in his field. His claims were embedded in pedagogical textbooks, most especially his own, and were taught to generations of teachers and administrators.

Teachers of schools of education willingly grasped Thorndike's fntstic ideas. Why?I think, because it gave them appearance of professionalism. Kolesnik writes on p. 7:

> Entirely too common in educational literature and discussions today are unqualified assertions to the effect that mental discipline is "no longer held." $< \cdots >$ Thousands of teachers must have been instructed, while seeking their certification, that the theory was completely disproved by Edward Lee Thorndike in the year 1901.

Just one example of Thorndike's influence today. The following problem may be used almost everywhere around the globe without objections:

Problem 55 Sally is five years older than her brother Bill. Four years from now, she will be twice as old as Bill will be then. How old is Sally now? $\boxed{\textbf{P : Sally}}$

In America it is declared unfit for the following reason: "First of all, who would ask such a question! Who would want to know this? If Bill and Sally can't figure it out, then this is some dumb family."[**Smith**], p. 85.

Notice that Smith does not refer to Thorndike or any other authority. He seems to think that what he says is just undeniable common sense. This is much worse than if he referred to somebody. This means that some part of American population, including Smith, is so completely brain-washed that cannot even imagine an alternative point of view.

As an example of an opposite, much more sound approach, let me quote one of Perelman's books [**Perelman.A**], where the second chapter, called "The language of algebra", consists of 25 sections, each devoted to a problem. One of them, called "An equation thinks for us", starts as follows:

Problem 56 If you doubt that an equation is sometimes more prudential than we are, solve the following problem: The father is 32 years old, the son is 5 years old. How many years later will the father's age be ten times the son's age? $\boxed{\textbf{P : age}}$

An equation is made and solved, but the answer is negative: -2. What does this mean? Perelman explains: "When we made the equation, we did not think that the father's age will never be ten times the son's age in the *future* - this relation could take place only in the *past*. The equation turned out more thoughtful and reminded us of our omission." I believe that this comment is really instructive, and constitutes a sufficient reason to discuss such a problem.

By the by, the problem 55 $\boxed{P : Sally}$ probably is classified in America as "algebra", but it easily can be solved arithmetically. It is sufficient to observe that the difference of ages is one and the same all the time. So four years from now Sally will be five years older than Bill and twice as old as Bill. So Bill's age will equal the difference of ages, which is five years. So Sally's age at that time will be twice this, that is ten years. So Sally is six years old now. In connection with the problem 27 $\boxed{P : Ber : age}$ I have quoted an advice how to solve an age problem in a Russian school book.

Arithmetical solutions are valued not only in Russia, but also in China. I remember having a group of American students of College Algebra patiently filling charts to solve a problem. One Chinese student did nothing. I asked if she had already solved the problem. She said, yes. I asked her to show her solution on the board and she wrote a one-line solution. All the others were astonished. But I could not have them solve problems in this style; I had tried it several times and we always lost our way in argufying about confused solutions. The method of charts allowed me to make students go step after step, so that every mistake was immediately localized and discussed.

Kolesnik and Ravitch quote only American sources and this is understandable, because otherwise they would never finish their work. However, Thorndike's ideas were criticised sharply abroad. In particular, Lev Vygotsky, a famous Russian psychologist, criticised Thorndike very thoroughly in his works [**Vygotsky.D, Vygotsky.T**]. Vygotsky wrote:

> As it is well-known, Thorndike, logically developing ideas, underlying his zoological experiments, came to a very definite theory of learning, which Koffka's book refutes quite thoroughly, thereby liberating us from the power of false and prejudiced ideas. [**Vygotsky.D**], P. 284.

The word "zoological" shows Vygotsky's anger, which I completely share. Indeed, Thorndike's theory treats human beings as analogs of animals reacting only to concrete stimuli, for whom a slight deviation from reality makes a problem irrelevant.

It seems that Thorndike's ideas serves some deep psychological needs of some American educators. They needed his ideas so badly that were willing even to misinterpret results of experiments to stick to his ideas. Ravitch refers to a book by Richard Hofstadter [**Hofst**], p. 349 and writes (p. 69):

> After reviewing this controversy, historian Richard Hofstadter concluded that "misuse of experimental evidence" by opponents of mental training "constitutes a major scandal in the history of educational thought."

Since its formation, the United States of America for a long time have been a symbol of democracy and liberty for the whole world. If this image is going to remain, we may expect American educators to teach all children efficiently. Regretfully, there are opposite tendencies in American education.

In view of Luria, Cole, Scribnet and Tulviste's studies, Thorndike and his followers wanted the school books to be filled only with such problems, which people from "traditional cultures" would be able to solve. I admit that even in industrial countries there may be individuals, families, even communities whose mentality is still on that "traditional" level. May be, children from such families need some special programs. But Thorndike and his followers want to foist such programs at **all** children or at least at majority of them!

30. Connections between school math and science

When I studied in a public school in Moscow, we studied physics for several years in the following order: mechanics, electricity, gases, opthics, atomic physics. All the time we solved problems which involved algebra, geometry, trigonometry and some thinking. This interaction of school subjects is very useful.

Every student was required to take all the main courses including mathematics and physics and the courses of mathematics and physics were always stronly correlated.

The course of physics was full of problems that needed algebra, geometry and trigonometry and the course of mathematics included some problems with physical content. This is an example:

Problem 57 A train started from a station and, moving with a constant acceleration, made a distance of 2.1 km and ended with a speed of 54 km/hour. Find the acceleration of the train and the time it spent. [**Larichev**], p. 257.

Such coordination of courses is possible only if there is a well-thought curriculum and a certain obligatory core. To apply trigonometry, the teacher of physics must be sure that all the students have taken it. In American schools you never can be sure about anything.

Considering American educators' obscession with "real world", one might expect them to care about connections between mathematics and physics. However, the reality is opposite. When American educators speak about "real world", they mean mostly trivial everyday needs and routine.

Chaos instead of curriculum leads to isolation of subjects from each other in American school, notably mathematics from physics. Many American school students never take physics and nobody tells them that they miss something important. Once, teaching a course of calculus, I solved a mechanical problem and then said: "The same result can be obtained from the law of conservation of energy." Silence. I asked: "Who has ever heard about the law of conservation of energy?" There was one foreign student in the group (from Taiwan) and he was the only one to raise his hand.

What is conspucious by its absence in American schools is application of mathematics at other lessons. The more my children studied in American schools, the more astonished I became to which extent are de-mathematicised even those courses where math is most appropriate.

An example. When my daughter was a sophomore in a "national exemplary" high school, she took "honors chemistry". The book was dreadful in the sense that it was absolutely dematheматized. The leaders of the independent school district, who approved this book, felt that something was missing, and created several leaflets with problems to compensate this. One of them contained the following problem:

Problem 58 What is wavelength of an electron of mass $9.11 \times 10^{-}28$ g traveling at 2.00×10^{8} m/s ? $\boxed{\textbf{P : electron}}$

I asked what the teacher said, my daughter answered: "She started as if she was going to explain something, but actually gave only a formula how to do it." This is understandable. Even if I were in that teacher's place, I also would be unable to explain it. My daughter was not dumb and she was reasonably interested in everything including science. She could study physics or chemistry at a resonable level, but nobody cared to teach her. She cannot find out how much time a stone takes to fall from the height of 100 meters and she cannot solve other elementary problems in physics. She never was taught laws of Newton. If she ever heard about laws of conservation, it is from me, not from the school. What about electron, I had told her that electron is a small particle with a negative electric charge. But how can a small particle have a wavelength ? Usully I tried to answer my daughter's questions, but this time I admitted a failure.

31. Guessing The Teacher's Mind

Another impression I got from helping my daughter with her homework is that she often had to guess the teacher's mind. This was especially often when the questions were not just computational, but involved what educational authorities probably expected to be questions "on understanding". While I was thinking, desperately trying to find a meaningful answer to a vague question, my daughter exclaimed "I know what she wants!" and wrote something which seemes meaningless to me, but got approval from her teacher. My daughter was good at guessing teachers' minds and always got good grades at school. Then she got good grades at a prestigious art college, graduated from it and now cannot find a decent job. My son, on the contrary, did not care to read teachers' minds. He and his high school hated each other. He got a GED (General Education Diploma), started to take college-level courses and got a well-paid position even before he got a college diploma. So much about preparation for real world in American schools.

32. Missing, Surplus or Contradictory Data

Several experiments have shown children's lack of comprehension of word problems using MSCD problemsw in a very dramatic form. The word “scandal” was used in this connection more than once.

“The Captain’s Age” and “The Shepherd’s age” are labels for a group of experiments conducted in France, Germany, Switzerland, Poland and perhaps some other countries. In his interesting article [**Selter**], Christoph Selter mentions the problem 50 $\boxed{P : captain}$ and then writes:

> Some French researchers then tested children aged 7 to 11 by using comparable problems - with similar, shattering results. The researchers let animals drop from the ship, which caused the children to subtract numbers that were not related; or they chose big and small numbers of animals, which caused the children to divide the numbers.

Then Selter tells about similar experiments conducted by himself and his colleagues with similar results in Germany. One of them was done in 1983 by Hendrik Radatz, who studied dependance of this phenomenon on age of children. He found that while only about 10 per cent of the Kindergarten children and first graders worked on insoluble problems, the percentage for second graders was about 30 per cent and the percentage for third and fourth graders was about 60 per cent, which is by far higher, and only in Grade 5 it goes down to 45 per cent. Radatz concludes that pupils’ behaviour is decisively influenced by the amount of mathematics teaching they have already received. The hypothesis seemed to be confirmed that “arithmetic ... is seen as a kind of play with artificial rules and without any particular link to reality ... The incompatibility of solutions with reality ... is not perceived by many primary pupils”.

Selter and his collaborators gave a group of children the following six problems:

- Michael is 8 years old. His mother is 26 years older than Michael. How old is she?

- Anke is 12 years old. Anke's mother is three times as old. How old is the mother?
- A shepherd owns 19 sheep and 13 goats. How old is the shepherd?
- A 27-year-old shepherd owns 25 sheep and 10 goats. How old is the shepherd?
- There are 13 boys and 15 girls sitting in a classroom. How old is the teacher?
- A bee-keeper has 5 bee-hives with 80 bees each. How old is the bee-keeper?

Then Selter writes:

> When we came together after the interviews, we were shocked. Each pupil interviewed had solved all six tasks by somehow connecting the given data. Our children had even added or subtracted to work out the fourth problem where the text plainly indicated that the shepherd was 27.

Selter also studied how this phenomenon depends on various conditions. He writes:

> There was a steadily increasing exchange of ideas in our group, and more widely at our institute, about the phenomena we had observed and the questions we had addressed. As a result, the six problems were given to quite a large number of third graders under slightly changed conditions. The results could be summarised as follows: text variations (numbers written as words, different contexts, changed sequence) did not have much effect; but if the initial contract made between the interviewer and the interviewee were changed, we could observe quite different results: for example, if the interview began with a reference to some problems being soluble and others being insoluble, fewer pupils tried to work out the latter.

There are other observations of this sort. Schoenfeld [**Schoenfeld.C**] tells about Reusser's experiments:

> Also he asked 101 fourth and fifth grade students to work the following problem: "Yesterday 33 boats sailed into the port and 54 boats left it. Yesterday at noon there were 40 boats left in the port. How many boats were still in the port yesterday evening?" $< \cdots >$ He reports thay all but one of the 101 students produced a numerical solution to the problem, and that only one student complained that the problem was ill-defined and unsolvable.

Any person, really interested in finding the answer, would notice that it is not said, before or after noon those boats entered or left the port. But different assumptions about it lead to different answers. In fact it is possible to obtain any number from zero to 73 making different assumptions about movements of boats. However, as I understand Schoenfeld's description, all Reusser's subjects presented just one number as the answer, although these numbers probably were different.

Thus, European (French, German, Suiss, Polish) educators are worried by children's willingness to tamper with data, which do not determine an answer. They want children to be critical and refuse to answer a question if no answer can be obtained in a rationl way.

Some European countries already approve inclusion of such problems into curricula with the intention to make children alert to them. An example:

> [I]n Poland, the official curriculum encourages teachers of grades 1-3 to give children occasionsl MSCD problems $< \cdots >$ [because] mathematics lessons should contribute to the development of the child's critical thinking.

I must admit that when I read this quote, I mentally agreed with the Polish decision.

I thought that such unusual, even somewhat jocular problems, should be used like pepper: a little bit everywhere. However, Puchalska and Semadeni, based on their observations, object to it as follows:

> However, our study shows tht MSCD problems can be effective only if several of them are given to pupils consecutively. Isolated MSCD problems cause bewilderment and should not be used till the pupils are accustomed to such problems. Proper explanation of the initial MSCD problems is very important, but still more important is that children should be encouraged to express their opinions and discuss the problems.

This argumentation reminded me that no recommendations should be issued without a solid prior investigation, both experimental and theoric. The responsible, thoughtful attitude of [**PS**], [**Selter**] and other European studies makes a contrast with self-assured edubabble of the 1989 “standards” and other educational documents issued by the most powerful American educators.

American educators also want to include such problems into curriculum, but without any experimentation and with opposite intentions! They want to put children at the mercy of their teachers. What will happen to an American school student if he refuses to tamper with a fuzzy situation included in an officially approved program? Evidently, he will get a bad grade, at least I have never seen an indication to the opposite in American “standards”, browsing which I found no usage of words “right”, “wrong”, “true”, “false” or their synonims. Instead of solving problems students are expected to “explore” them, instead of proving theorems, students are expected to “discover” them. How will teachers decide who gets better grades after such “explorations” and “discoveries”? Evidently, those who please their teachers most. In an American school you get a good grade not when you are right, but when you guess the teacher’s mind.

In fact, "the Captain's Age" was part of jocular folklore already in my childhood. I was only a few years old when one of my relatives (I forgot, who) played a joke with me. He said: "Imagine that you are captain of a ship. There are nine saylors and fifteen passangers aboard. Also there are sixty seven rats and a hundred twenty eight cockroaches. What is the captain's name?" I answered that it was impossible to determine the captain's name, and then the relative (probably, it was an older cousin, an adult would not be so happy) triumphantly reminded me: "Remember with what I started: *Imagine that* *you* *are captain of a ship.* So the captain's name is Andrei! In the subsequent years I played this joke with younger children several times.

Ya. S. Dubnov, a prominent Russian educator, used this joke to make a profound comparison:

> Finally, examples will be given of proofs whose invalidity stems from the fact that the proposition asserted has nothing to do with the given data. How this may come about I shall attempt to explain by an example which is remote from geometry and science in general.
>
> The following facetious problem is well-known: "A steamer is situated at latitude $42^o\ 15'\ N$ and longitude $17^o\ 32'\ W$. [The figures are taken at random; usually further data is added which complicate the conditions.] How old is the captain?" For our purpose let us alter the question of the problem somewhat. "Is the assertion correct that the captain is more than 45 years old?" It is clear to everyone that it is impossible to draw such a conclusion from the data given in the conditions of the problem, and that any attempt to prove the assertion concerning the age of the captain is destined to end in failure. Moreover, it is possible to *prove* that it is impossible to prove the assertion. Actually the steamship company, about which we learn nothing from the data of the problem, may chart a course which passes through the

geographic point indicated and assign to the voyage a captain of this or that age, assuming that the company has captains of any age available for such trips.

$< \cdots >$

Returning more nearly to our subject, let us ask "Is it true that the sum of the angles of any triangle is equal to two right angles?" Every schoolboy who has studied the chapter on parallel straight lines in a geometry textbook is acquainted with the proof of this important theorem, but few know its history, which goes back 2,000 years. The proof is based on the properties of angles formed by a line intersecting parallel straight lines, and these properties are based in turn on the so-called "parallel postulate": *Only* one straight line can bedrawn parallel to a given line through a given point not on the line.

$< \cdots >$

It was not until the eifghteen-twenties that the Russian mathematician Nikolai Lobachevskii (1792-1856) discovered the cause for the failure of all attempts to prove the parallel postulate. He constructed an extensive and profound theory of geometry, of which I shall not attempt to give here even the remotest idea.

$< \cdots >$

However complex the theory of Lobachevskii, and on the other hand, however naive the problem about the age of the captain, the "proof of the impossibility of the proof" is of the same nature in both problems. In these problems we are given certain data, and we desire to show that a certain conclusion cannot be logically deduced from the data. To do this we find concrete examples (called "models") in which all the given conditions are satisfied, but in some of which the conclusion in question is true and in others the conclusion is false. [**Dubnov.M**], p. 3.

Let us not reproach Dubnov for that jingoism with which he failed to mention Bolyai and Gauss. This is not Dubnov's fault, but rather result of those conditions in which he had to publish and survive. (The book was first published in Stalin's lifetime.) Let us pay attention to the fact that publiction of this comparison "legalized" answers "this cannot be done" in school and even elevated such answers to principial height.

33. Didactic Contract

The phrase *didactic contract* has been mentioned in connection with these observations many times. Indeed, whenever we find ourselves in a certain situation, we obey certain rules of behavior. For example, whenever I fly with a plane, I try not to conflict with the crew and obey the rules set by them.

Analogously, when children come to school, they immediately notice that this is a special place with certain rules of behavior different from all places where they had been before. Some of these rules seem quite normal for children like "don't break windows", but some others are more difficult for vivid children to comply with, for example "don't stand up during a lesson".

Then a lesson starts and children fulfill teachers' assignments, which also have to be done according to certain rules. The school's initial purpose was to prepare children to deal with relity, but school inevitably becomes a reality of its own kind and children have to comply with it.

Indeed, children who solved unsolvable problems, evidently, thought that this was what powerful adults wanted them to do. Evidently, they misunderstood the message their teachers had sent to them. Evidently, this message was not clear enough.

Selter indicates that the best way to improve children's performance was just to tell them that some problems may be unsolvable. This makes me think that the true cause of this phenomenon from the very beginning also what was or was not told the students.

If children confuse the didactic contract, it makes sense to look whether teachers don't do the same.

One of the most irritating qualities of bad teachers is irrelevant pedantry.

An example. Once, when I taught a group of teachers or future teachers in America, we solved a problem in two ways and obtained $\sqrt{2}$ in one case and $2/\sqrt{2}$ in the other. "Which of these answers is false?" – asked I as a provocation. I hoped for an answer: "Both are correct. Don't you see that they are equal?" However, the answer issued by one student and supported by all the others was: "The second is false because we may not divide by a root." This made me sad. It is true that we should prefer expressions without roots in denomintors. But it is just a small convenience, much less important than the fact that $\sqrt{2} = 2/\sqrt{2}$!

Teachers of small competence encumber the didactic contract with lots of petty rules, which makes their students lose from sight the most important principles.

Another example. Several school teachers took my course of Abstract Algebra. I explained that algebraic equations of degree 5 and more generally cannot be solved in radicals, but some special ones can. As a homework I gave an equation of a high degree and assigned to solve it and thereby to show that it can be solved in radicals. All my students successfully factorized the polynom and found all the roots, but were not satisfied with it and presented decimal approximations of these roots as a final answer. In schools of education they were told repeatedly that an expression involving roots is not yet an answer!

34. Mathematical Education and Morality

In addition to general human ethics, many professions need special ethics to solve special problems faced by those who practice these professions. There is medical ethics (whether or under which conditions euthanasy, assisted suicide, usage of human subjects or abortion are permissible), military ethics (who and under which conditions may kill whom), literary ethics (what is plagiarism) and many others. Successful doing mathematics needs a certain kind of "mathematical ethics", which develops along with mathematical competence. To do mathematics, one needs a special "honesty of mind", which is not needed and does not develop in plain everyday life. Ya. S. Dubnov, a prominent Russian educator, wrote in a referee's report about a textbook:

> Now I can formulate,what exactly is unacceptable for me in this textbook: it teaches in the best case, but does not morally ciltivate. But it is necessary to cultivate not only independence of thinking, but also that "mathematical morality", which forbids to pronounce shallow words, devoid of exact meaning and to present as proved what in fact is only hinted at. [**Dubnov.T**], p. 227

However important is development of children's abilities for exact and abstract mathematical thinking, we should not expect that this development will automatically raise better persons. I would rather say that abstract thinking and common morality are "orthogonal" to each other in the sense that presence or absence of each may combine with presence or absence of the other. One sad consequence of this is intelligent crime. This is an illustration [**hackers**]:

> *MOSCOW – Young, smart Russian hackers are posing an increasing threat to global business, police said on Wednesday following last week's arrest of an online extortion ring that cost British companies up to $70 million.*
>
> *Russia, with its highly educated workforce and inefficient police, has become infamous for computer piracy and crime.*

I hate to quote it, but it is an important warning: if hackers have well-developed formal thinking, honest people need at least the same.

35. Conclusion

Having read this article, somebody may think that I connect poor education with democracy. I do not. History connects them sometimes, at random as it seems. What I do think is that democracy has many aspects and free elections of political leaders is just one of them. Quality of education is not determined by quality of political structure and can deviate from it for better or worse. Let me illustrate this idea by two examples. There are 2354 problems in [**Berez**], a few dozens of which contain Soviet political propaganda. This is an example:

Problem 59 How many years passed from the French burgeois revolution till the Great October socialist revolution if the former took place in 1789? (p. 17). $\boxed{\textbf{P : revolution}}$

Here the political bias is evident, but from the mathematical point of view the problem is fair: the correct answer is arguably $1917 - 1789 = 128$. Soviet leaders wanted such problems to indoctrinate Soviet ideology along with teaching mathematics, but only half of their wishes came true: students learned mathematics, but got rid of the Soviet rule. Some rushed to the West taking jobs from those who were raised on problems like this:

Problem 60 A national magazine surveyed teenagers to determine the number of hours of TV they watched each day. How many hours do you think the magazine reported? [**St.1**], p. 79. $\boxed{\textbf{P : TV}}$

It may seem very human to use this problem in class: every student will be able to say something and nobody will be completely wrong, so nobody will be frustrated. But when these kids grow up, they will regret the years wasted on such shallow “problems”.

Let us read Berriozábal again:

> We must condemn those educational programs and reforms that would substitute the mere acquisition of computer manipulative skills and access to the Internet for intellectual development. We must not stifle our children's educational development with Fuzzy Math programs that emphasize process over content. $< \cdots >$
>
> Today, with the current education fads, our students can't read – instead stories and instructions are given on user friendly videotapes; they can't write an error free sentence unless they have acess to spell checkers; and they can't do basic computational work unless they have a calculator. $< \cdots >$
>
> Consequently, many of these students through no fault of their own, but rather the fault of our educational system, aredoomed to enter dull, entry level jobs with basic high tech applications. Rather than being our future educated productive leaders in our high tech society, their managers and supervisors will be well educated engineers and scientists who are imported from other countries.

I hope that pretty soon many American parents will say to teachers of their children: "Stop pretending to do the impossible, but do the possible by hell! Don't pretend to teach our children fractals, but teach them fractions properly! Don't pretend to teach them non-Euclidean geometry, but teach them Euclidean geometry with proofs! Don't pretend to teach them Student's and chi-square distributions, but teach them elementary algebra with problems so that they would not complain on tests "we did not solve such problems"! Don't pretend to teach them phantoms like "real-world problems" but teach them to apply elementary mathematics to physics and computer science!" This will put American education on a much better place in the world competition.

What is democracy in education? Let me mention one important parameter of it: students should be allowed to study objective reality rather than fads of educational leaders or adjust to fancies and guess minds of their teachers. The most important purpose of mathematical education, as I see it, is to bring the students to such a level at which everyone of them can say to the teacher: “Now I can decide what is true and what is false and don’t need you to tell me this anymore.” Regretfully, all the aspirations of “reformers” of American mathematical education go in the opposite direction. Mastery of algorithms makes students self-sufficient – get away from it, make them dependent on Texas Instruments. Logical proofs develop students’ mental discipline – get away from them also. Traditional word problems allow to determine the right answer – get away from them also. What the “reformers” promote, that is open-ended problems, “real-world” problems with clouds of answers, activities instead of problems, create a fuzzy world, in which students always are at the teacher’s mercy and can not learn to discriminate between right and wrong by their own means. How does this combine with the traditional American values of integrity and independence? I believe, it does not and in the long run one will prevail and the other disappear.

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