

# The Best Possible Unification for Any Collection of Physical Theories\*

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*Abstract:* It is shown that the set of all finitary consequence operators defined on any nonempty language is a join-complete lattice. This result is applied to various collections of physical theories to obtain an unrestricted supremum unification.

## 1. Introduction.

In Herrmann (2001a, b), a restricted hyperfinite ultralogic unification is constructed. The restrictions placed upon this construction were necessary in order to relate the constructed ultralogic directly to the types of ultralogics used to model probability models (Herrmann 2001c, d). In particular, the standard collections of consequence operators are restricted to a very special set of operators  $H_X$ , where  $X$  is itself restricted to the set of all significant members of a language  $\Lambda$ . In this paper, all such restrictions are removed. For reader convince, some of the introductory remarks that appear in Herrmann (2001a, b) are repeated. Over seventy years ago, Tarski (1956, pp. 60-109) introduced *consequence* operators as models for various aspects of human thought. There are two such mathematical theories investigated, the *general* and the *finitary* consequence operators (Herrmann, 1987). Let  $L$  be a nonempty language,  $\mathcal{P}$  be the power set operator and  $\mathcal{F}$  the finite power set operator. There are three cardinality independent axioms.

**Definition 1.1.** A mapping  $C: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$  is a general consequence operator (or closure operator) if for each  $X, Y \in \mathcal{P}(L)$

- (1)  $X \subset C(X) = C(C(X)) \subset L$ ; and if
- (2)  $X \subset Y$ , then  $C(X) \subset C(Y)$ .

A consequence operator  $C$  defined on  $L$  is said to be *finitary* (*finite*) if it satisfies

- (3)  $C(X) = \bigcup \{C(A) \mid A \in \mathcal{F}(X)\}$ .

**Remark 1.2.** The above axioms (1), (2), (3) are not independent. Indeed, (1) and (3) imply (2). Clearly, the set of all finitary consequence operators defined on a specific language is a subset of the set of all general operators. The phrase “defined on  $L$ ” means formally defined on  $\mathcal{P}(L)$ .

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\*Any typographical errors that appear in the published version of this paper are caused by faulty publisher editing.

All known scientific logic-systems use finitely many rules of inference and finitely many steps in the construction of a deduction from these rules. Hence, as shown in Herrmann (2001a, b), the consequence operator that models such theory generating thought processes is a finitary consequence operator. Although many of the results in this paper hold for the general consequence operator, we are only interested in collections of finitary consequence operators. Dziobiak (1981, p. 180) states the Theorem 2.10 below. However, the statement is made without a formal proof and is relative to a special propositional language. Theorem 2.10 is obtained by using only basic set-theoretic notions and Tarski's basic results for any language. Further, the proof reveals some interesting facts not previously known. Unless noted, all utilized Tarski (1956, pp. 60-91) results are cardinality independent.

## 2. The Lattice of Finitary Operators.

**Definition 2.1.** In all that follows, any set of consequence operators will be nonempty and each is defined on a nonempty language. Define the relation  $\leq$  on the set  $\mathcal{C}$  of all general consequence operators defined on  $L$  by stipulating that for any  $C_1, C_2 \in \mathcal{C}$ ,  $C_1 \leq C_2$  if for every  $X \in \mathcal{P}(L)$ ,  $C_1(X) \subset C_2(X)$ .

Obviously,  $\leq$  is a partial order contained in  $\mathcal{C} \times \mathcal{C}$ . Our standard result will show that for the entire set of finitary consequence operators  $\mathcal{C}_f \subset \mathcal{C}$  defined on  $L$ , the structure  $\langle \mathcal{C}_f, \leq \rangle$  is a lattice.

**Definition 2.2.** Define  $I: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$  and  $U: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$  as follows: for each  $X \subset L$ , let  $I(X) = X$ , and let  $U(X) = L$ .

Notice that  $I$  is the lower unit (the least element) and  $U$  the upper unit (the greatest element) for  $\langle \mathcal{C}_f, \leq \rangle$  and  $\langle \mathcal{C}, \leq \rangle$ .

**Definition 2.3.** Let  $C \in \mathcal{C}$ . A set  $X \subset L$  is a  $C$ -system or simply a system if  $C(X) \subset X$  and, hence, if  $C(X) = X$ . For each  $C \in \mathcal{C}$ , let  $\mathcal{S}(C) = \{X \mid (X \subset L) \wedge (C(X) = X)\}$ .

Since  $C(L) = L$  for each  $C \in \mathcal{C}$ , then each  $\mathcal{S}(C) \neq \emptyset$ .

**Lemma 2.4** *For each  $C_1, C_2 \in \mathcal{C}$ ,  $C_1 \leq C_2$  if and only if  $\mathcal{S}(C_2) \subset \mathcal{S}(C_1)$ .*

*Proof.* Let any  $C_1, C_2 \in \mathcal{C}$  and  $C_1 \leq C_2$ . Consider any  $Y \in \mathcal{S}(C_2)$ . Then  $C_1(Y) \subset C_2(Y) = Y$ . Thus,  $C_1 \in \mathcal{S}(C_1)$  implies that  $\mathcal{S}(C_2) \subset \mathcal{S}(C_1)$ .

Conversely, suppose that  $\mathcal{S}(C_2) \subset \mathcal{S}(C_1)$ . Let  $X \subset L$ . Then since, by axiom 1,  $C_2(X) \in \mathcal{S}(C_2)$ , it follows, from the requirement that  $C_2(X) \in \mathcal{S}(C_1)$ , that  $C_1(C_2(X)) = C_2(X)$ . But  $X \subset C_2(X)$  implies that  $C_1(X) \subset C_1(C_2(X)) = C_2(X)$ , from axiom 2. Hence,  $C_1 \leq C_2$  and the proof is complete.

**Definition 2.5.** For each  $C_1, C_2 \in \mathcal{C}$ , define the following binary relations in  $\mathcal{P}(L) \times \mathcal{P}(L)$ . For each  $X \subset L$ , let  $(C_1 \wedge C_2)(X) = C_1(X) \cap C_2(X)$  and  $(C_1 \vee_w C_2) = \bigcap \{Y \subset L \mid (X \subset Y = C_1(Y) = C_2(Y))\}$  For finitely many members of  $\mathcal{C}$ , the operators  $\wedge, \vee_w$  are obviously commutative and associative. These two relations are extended to arbitrary  $\mathcal{A} \subset \mathcal{C}$  by defining  $(\bigwedge \mathcal{A})(X) = \bigwedge \mathcal{A}(X) = \bigcap \{C(X) \mid C \in \mathcal{A}\}$

and  $(\bigvee_w \mathcal{A})(X) = \bigvee_w \mathcal{A}(X) = \bigcap\{Y \subset L \mid X \subset Y = C(Y) \text{ for all } C \in \mathcal{A}\}$  (Dziobiak, 1981, p. 178). Notice that  $\bigvee_w \mathcal{A}(X) = \bigcap\{Y \subset L \mid (X \subset Y) \wedge (Y \in \bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\})\}$ .

**Lemma 2.6.** *Let  $\mathcal{A} \subset \mathcal{C}$  [resp.  $\mathcal{C}_f$ ] and  $\mathcal{S}' = \{X \mid (X \subset L) \wedge (X = \bigvee_w \mathcal{A}(X))\}$ . Then  $\mathcal{S}' = \bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$ .*

*Proof.* By Tarski's Theorem 11 (b) (1956, p. 71), which holds for finitary and general consequence operators, for each  $X \subset L$  and  $C \in \mathcal{A}$ ,  $X \subset \bigvee_w \mathcal{A}(X) = Y' \in \mathcal{S}(C)$ . Hence, if  $Y' \in \mathcal{S}'$ , then  $\bigvee_w \mathcal{A}(Y') = Y' \in \mathcal{S}(C)$  for each  $C \in \mathcal{A}$ . Thus  $\mathcal{S}' \subset \bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$ . Conversely, let  $Y \in \bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$ . From the definition of  $\bigvee_w$ ,  $\bigvee_w \mathcal{A}(Y) = Y$  and, hence,  $Y \in \mathcal{S}'$  and this completes the proof.

**Lemma 2.7.** *Let nonempty  $\mathcal{B} \subset \mathcal{P}(L)$  and  $L \in \mathcal{B}$ . Then the operator  $C_{\mathcal{B}}$  defined for each  $X \subset L$  by  $C_{\mathcal{B}}(X) = \bigcap\{Y \mid X \subset Y \in \mathcal{B}\}$  is a general consequence operator defined on  $L$ .*

*Proof.* Assuming the hypothesis, it is obvious that  $C_{\mathcal{B}}: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$  and  $X \subset C_{\mathcal{B}}(X)$ . Clearly, if  $Z \subset X \subset L$ , then  $C_{\mathcal{B}}(Z) \subset C_{\mathcal{B}}(X)$ ; and, for each  $Y \in \mathcal{B}$ ,  $X \subset Y$  if and only if  $C_{\mathcal{B}}(X) \subset Y$ . Hence,  $C_{\mathcal{B}}(C_{\mathcal{B}}(X)) = \bigcap\{Y \mid C_{\mathcal{B}}(X) \subset Y \in \mathcal{B}\} = C_{\mathcal{B}}(X)$ . This completes the proof.

**Remark 2.8.** The hypothesis of Lemma 2.7 is not restricted to a collection that is closed under arbitrary intersection.

**Theorem 2.9.** *With respect to the partial order relation  $\leq$  defined on  $L$ , the structure  $\langle \mathcal{C}, \bigvee_w, \wedge, I, U \rangle$  is a complete lattice with upper and lower units.*

*Proof.* Let  $\mathcal{A} \subset \mathcal{C}$  and  $\mathcal{B} = \bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$ . Since  $L \in \mathcal{B}$ , then by Lemma 2.7,  $\bigvee_w \mathcal{A} = C_{\mathcal{B}} \in \mathcal{C}$ . Moreover, by Lemmas 2.4 and 2.6,  $C_{\mathcal{B}}$  is the least upper bound for  $\mathcal{A}$  with respect to  $\leq$ .

Next, let  $\mathcal{B} = \bigcup\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$ . For  $X \subset L$ ,  $X \subset C(X)$  for each  $C \in \mathcal{A}$ . For each  $C \in \mathcal{A}$ , there does not exist a  $Y_C$  such that  $Y_C \in \mathcal{S}(C)$ ,  $X \neq Y_C$ ,  $Y_C \neq C(X)$  and  $X \subset Y_C \subset C(X)$ . Hence,  $C_{\mathcal{B}}(X) = \bigcap\{Y \mid X \subset Y \in \mathcal{B}\} = \bigcap\{C(X) \mid C \in \mathcal{A}\} = \bigwedge \mathcal{A}(X)$ . Hence,  $\bigwedge \mathcal{A} \in \mathcal{C}$  and it is obvious that  $\bigwedge \mathcal{A}$  is the greatest lower bound for  $\mathcal{A}$  with respect to  $\leq$ . This completes the proof.

Although the proof appears in error, (Wójcicki, 1970) stated Theorem 2.9 for a propositional language. In what follows, we only investigate the basic lattice structure for  $\langle \mathcal{C}_f, \leq \rangle$ .

**Theorem 2.10.** *With respect to the partial order relation  $\leq$  defined on  $\mathcal{C}_f$ , the structure  $\langle \mathcal{C}_f, \bigvee_w, \wedge, I, U \rangle$  is a lattice with upper and lower units.*

*Proof.* It is only necessary to consider two distinct  $C_1, C_2 \in \mathcal{C}_f$ . As mentioned, the commutative and associative laws hold for  $\wedge$  and  $\bigvee_w$  and by definition each maps  $\mathcal{P}(L)$  into  $\mathcal{P}(L)$ . In  $\langle \mathcal{C}, \leq \rangle$ , using theorem 2.9, axiom 1 and 2 hold for the greatest lower bound  $C_1 \wedge C_2$  and for the least upper bound  $C_1 \bigvee_w C_2$ . Next, we have that  $(C_1 \wedge C_2)(X) = (\bigcup\{C_1(Y) \mid Y \in \mathcal{F}(X)\}) \cap (\bigcup\{C_2(Y) \mid Y \in \mathcal{F}(X)\}) = \bigcup\{C_1(Y) \cap C_2(Y) \mid Y \in \mathcal{F}(X)\} = \bigcup\{(C_1 \wedge C_2)(Y) \mid Y \in \mathcal{F}(X)\}$  and axiom 3 holds and, hence,  $C_1 \wedge C_2 \in \mathcal{C}_f$ . Therefore,  $\langle \mathcal{C}_f, \wedge, I, U \rangle$  is, at the least, a meet semi-lattice.

Next, we show by direct means that for each  $C_1, C_2 \in \mathcal{C}_f$ ,  $C_1 \vee_w C_2 \in \mathcal{C}_f$ . Let (the cardinality of  $L$ )  $|L| = \Delta$ . For each  $X_i \subset L$ , ( $i \in \Delta$ ), let  $\mathcal{A}'(X_i) = \{Y \mid (X_i \subset Y \in \mathcal{S}(C_1) \cap \mathcal{S}(C_2)) \wedge (Y \subset L)\}$ . Let  $\bigcap\{Y \mid Y \in \mathcal{A}'(X_i)\} = Y_i$ . By Tarski's Theorem 11a (1956, p. 71),  $X_i \subset Y_i \in \mathcal{S}(C_1) \cap \mathcal{S}(C_2)$ , and by definition  $Y_i = (C_1 \vee_w C_2)(X_i)$ . Hence,  $Y_i \in \mathcal{A}'(X_i)$  and is the least ( $\subset$ ) element. For  $X_i \subset L$ , let  $\mathcal{A}''(X_i) = \{Y \mid (C_1(X_i) \subset Y \in \mathcal{S}(C_1) \cap \mathcal{S}(C_2)) \wedge (Y \subset L)\}$ . Since  $X_i \subset C_k(X_i)$ ,  $k = 1, 2$ , then  $\mathcal{A}'' \subset \mathcal{A}'$ . Since  $L \in \mathcal{A}'(X_i)$ ,  $\mathcal{A}'(X_i) \neq \emptyset$ . Indeed, let  $Y \in \mathcal{A}'(X_i)$ . Then  $X_i \subset C_k(Y) = Y$ ,  $k = 1, 2$ . Additionally,  $X_i \subset C_1(Y) = Y$  implies that  $X_i \subset C_1(X_i) = C_1(C_1(X_i)) \subset C_1(C_1(Y)) = C_1(Y) = Y$ . Hence, it follows that for any  $X_i \subset L$ ,  $\mathcal{A}''(X_i) = \mathcal{A}'(X_i)$ . For fixed  $X_i \subset L$ , let  $X_j \in \mathcal{F}(X_i)$ . Let  $Y_j$  be defined as above and, hence,  $Y_j$  is the least element in  $\mathcal{A}'(X_j) = \mathcal{A}''(X_j)$ . Consider  $\mathcal{D} = \{Y_j \mid X_j \in \mathcal{F}(X_i)\}$ , and, for  $j = 1, \dots, n$ , consider  $Y_j \in \mathcal{D}$  and the corresponding  $X_j \subset L$ . Let  $X_k = \bigcup\{X_j \mid j = 1, \dots, n\} \in \mathcal{F}(X_i)$ . Then  $Y_k = \bigcap\{Y \mid Y \in \mathcal{A}'(X_k)\} \in \mathcal{D}$ . If  $Y \in \mathcal{A}'(X_k)$ , then  $Y \in \mathcal{A}'(X_j)$ ,  $j = 1, \dots, n$ . Hence,  $Y_j \subset Y_k$ ,  $j = 1, \dots, n$  implies that  $Y_1 \cup \dots \cup Y_n \subset Y_k$ . Tarski's Theorem 12 (1956, p. 71) implies that  $Y^* = \bigcup\{Y_j \mid X_j \in \mathcal{F}(X_i)\} \in \mathcal{S}(C_1) \cap \mathcal{S}(C_2)$ . Also, by definition, for all  $X_j \subset L$ ,  $Y_j \in \mathcal{A}''(X_j)$  implies that  $C_1(X_j) \subset Y_j$ . The fact that  $C_1$  is finitary yields  $C_1(X_i) \subset Y^*$ . Hence,  $Y^* \in \mathcal{A}''(X_i)$ . Since  $C_1(X_j) \subset C_1(X_i)$ ,  $X_j \in \mathcal{F}(X_i)$ , then  $\mathcal{A}''(X_i) \subset \mathcal{A}''(X_j)$ . Thus  $Y_j \subset Y_i$ ,  $X_j \in \mathcal{F}(X_i)$ . Therefore,  $Y^* \subset Y_i$ . But,  $Y^* \in \mathcal{A}''(X_i)$  implies that  $Y^* = Y_i$ . Re-stating this last result,  $\bigcup\{(C_1 \vee_w C_2)(X_j) \mid X_j \in \mathcal{F}(X_i)\} = (C_1 \vee_w C_2)(X_i)$  and, therefore, axiom (3) holds for the binary relation  $\vee_w$  and  $\langle \mathcal{C}_f, \vee_w, \wedge, I, U \rangle$  is a lattice. This completes the proof.

**Corollary 2.10.1.** *Let each member of  $\mathcal{C}_f$  be defined on  $L$ . The structure  $\langle \mathcal{C}_f, \vee_w, \wedge, I, U \rangle$  is a join-complete lattice.*

Proof. Let  $\emptyset \neq \mathcal{A} \subset \mathcal{C}_f$ . Now simply modify the second part of the proof of Theorem 2.10 by substituting  $\bigcap\{\mathcal{S}(C) \mid C \in \mathcal{A}\}$  for  $\mathcal{S}(C_1) \cap \mathcal{S}(C_2)$  and this complete the proof.

**Remark 2.11.** Tarski's Theorem 12 used above requires his Theorem 4 and Theorem 4 requires that the consequence operators be finitary. Corollary 2.10.1 should be identical with Corollary 2.11 in Herrmann (2004). Unfortunately, various corrections to this published version were not made by the editor. It is known, since  $I$  is a lower bound for any  $\mathcal{A} \subset \mathcal{C}_f$ , that  $\langle \mathcal{C}_f, \vee_w, I, U \rangle$  is actually a complete lattice with a meet operator generated by the  $\vee_w$ -operator. It appears that the meet operator  $\wedge$  for infinite  $\mathcal{A}$  need not correspond, in general, to the  $\vee_w$  defined meet operator. Wójcicki [10] has constructed, for a set of consequence operators  $\mathcal{C}'$ , an infinite  $\mathcal{A} \subset \mathcal{C}'$  of finitary consequence operators, with some very special properties. However, the general consequence operator defined for each  $X \subset L$  by  $\bigcap\{C(X) \mid C \in \mathcal{A}\}$  is not a finitary operator. Thus, in general,  $\langle \mathcal{C}_f, \vee_w, \wedge, I, U \rangle$  need not meet-complete lattice. This behavior is not unusual. For example, let infinite  $X$  have an infinite topology  $\mathcal{T}$ . Then  $\langle \mathcal{T}, \cup, \cap, \emptyset, X \rangle$  is a join-complete sublattice of the lattice  $\langle \mathcal{P}(X), \cup, \cap, \emptyset, X \rangle$ . The structure  $\langle \mathcal{T}, \cup, \emptyset, X \rangle$  is actually complete, but it is not a meet-complete sublattice of complete  $\langle \mathcal{P}(X), \cup, \cap, \emptyset, X \rangle$ .

### 3. System Consistent Logic-systems

Let  $\Sigma$  be a non-empty set of science-community logic-systems and let  $|\cdot|$  denote cardinality. In practice,  $|\Sigma| \leq \aleph_0$ . Each logic-system  $S_i \in \Sigma$ ,  $i \in |\Sigma|$ , is defined on a countable language  $L_i$  and each  $S_i$  determines a specific finitary consequence operator  $C_i$  defined on a language  $L_i$ . At the least, by application of the insertion of hypotheses rule (Herrmann, 2001a/b, p. 94/2) for nonempty cardinal  $\Delta \leq |\Sigma|$ , each member of  $\{C_i \mid i \in \Delta\}$  is defined on the language  $\bigcup\{L_i \mid i \in \Delta\}$ . *In all that follows, a specific set of logic-system generated consequence operators  $\{C_i \mid i \in \Delta\}$  defined on a specific set of languages  $\{L_i \mid i \in \Delta\}$  will always be considered as trivially extended and, hence, defined by the insertion of hypotheses rule on the set  $\bigcup\{L_i \mid i \in \Delta\}$ .* In general, such a specific set of consequence operators is contained in the lattice of all finitary operators defined on  $\bigcup\{L_i \mid i \in \Delta\}$ . A logic-system  $S'$  and its corresponding consequence operator is a *trivial extension* of a logic-system's  $S$  defined on  $L$  where, for a language  $L' \supset L$ ,  $S'$  is the same as  $S$  except that only the hypotheses insertion rule is applied to  $L' - L$ . The system  $S'$  and its corresponding consequence operator  $C'$  is a non-trivial extension if it is extended to  $L'$  by insertion and some other n-ary relations that contain members of  $L' - L$  are adjoined to those in  $S$  or various original n-ary relations in  $S$  are extended by adding n-tuples that contain members from  $L' - L$ . For both the trivial and non-trivial cases and with respect to the language  $L'$ , it follows that  $C \leq C'$ . In the trivial case, if  $X \subset L'$ , then  $C'(X) = C(X \cap L) \cup (X - L)$ .

In practice, a *practical logic-system* is a logic-system defined for the subsets of a finite language  $L^f$ . When a specific deduction is made from a set of hypotheses  $X$ , the set  $X$  is finite. If the logic-system also includes 1-ary sets, such as the logical or physical axioms, the actual set of axioms that might be used for a deduction is also finite. Indeed, the actual set of all deductions obtained at any moment in human history and used by a science-community form a finite set of statements that are contained in a finite language  $L^f$ . (Finite languages, the associated consequence operators and the like will usually be denoted by a  $f$  superscript.) The finitely many n-ary relations that model the rules of inference for a practical logic-system are finite sets. Practical logic-systems generate practical consequence operators and practical consequence operators generate effectively practical logic-systems, in many ways. For example, the method found in Łoś, J. and R. Suszko (1958), when applied to a  $C^f$ , will generate effectively a finite set of rules of inference. The practical logic-system obtained from such rules generates the original practical consequence operator. Hence, a consequence operator  $C^f$  defined on  $L^f$  is considered a *practical consequence operator* although it may not correspond to a previously defined scientific practical logic-system; nevertheless, it does correspond to an equivalent practical logic-system.

Our definition of a *physical theory* is a refinement of the usual definition. Given a set of physical hypotheses, general scientific statements are deduced. If accepted by a science-community, these statements become *natural laws*. These natural laws then become part of a science-community's logic-system. In Herrmann (2001a, b,

a), a consequence operator generated by such a logic-system is denoted by  $S_N$ . From collections of such logic-systems, the  $S_N$  they generate are then applied to specific natural-system descriptions  $X$ . For scientific practical logic-systems, the language and rules of inference need not be completely determinate in that, in practice, the language and rules of inference are extended.

The complete Tarski definition for a consequence operator includes finite languages (1956, p. 63) and all of the Tarski results used in this paper apply to such finite languages. Theorem 2.10 holds for any language finite or not. In the lattice of finitary consequence operators defined on  $L^f$ ,  $\vee_w$  determines the least upper bound for a finite set of such operators. However, it is certainly possible that this least upper bound is the upper unit  $U$ .

**Definition 3.1.** Let  $C$  be a general consequence operator defined in  $L$ . Let  $X \subset L$ .

- (i) The set  $X$  is *C-consistent* if  $C(X) \neq L$ .
- (ii) The set  $X$  is *C-complete* if for each  $x \in L$ , either  $x \in X$  or  $C(X \cup \{x\}) = L$ .
- (iii) A set  $X \subset L$  is *maximally C-consistent* if  $X$  is *C-consistent* and whenever a set  $Y \neq X$  and  $X \subset Y \subset L$ , then  $C(Y) = L$ .

Notice that if  $X \subset L$  is *C-consistent*, then  $C(X)$  is a *C-consistent* extension of  $X$  which is also a *C-system*. Further, *C-consistent*  $W$  is *C-consistent* with respect to any trivial extension of  $C$  to a language  $L' \supset L$ .

**Theorem 3.2** *Let general consequence operator  $C$  be defined on  $L$ .*

(i) *The set  $X \subset L$  is C-complete and C-consistent if and only if  $X$  is a maximally C-consistent.*

(ii) *If  $X$  is maximally C-consistent, then  $X$  is a C-system.*

Proof. (i) Let  $X$  be maximally *C-consistent*. Then  $X$  is *C-consistent* and, hence,  $C(X) \neq L$ . Hence, let  $x \in L$  and  $x \notin X$ . Then  $X \subset X \cup \{x\}$  implies that  $X \cup \{x\}$  is not *C-consistent*. Thus  $C(X \cup \{x\}) = L$ . Hence,  $X$  is *C-complete*. Conversely, assume that  $X$  is *C-consistent* and *C-complete*. Then  $X \neq L$ . Let  $X \subset Y \subset L$  and  $X \neq Y$ . Hence, there is some  $y \in Y - X$  and from *C-completeness*  $L = C(X \cup \{y\}) \subset C(Y)$ . Thus,  $Y$  is not *C-consistent*. Hence,  $X$  is maximally *C-consistent* and the result follows.

(ii) From *C-consistency*,  $C(X) \neq L$ . If  $x \in C(X) - X$ , then maximally *C-consistent* implies that  $L = C(X \cup \{x\}) \subset C(C(X)) = C(X)$ . This contradiction yields that  $X$  is a *C-system*.

The following easily obtained result holds for many types of languages (Tarski, 1956, p. 98. Mendelson, 1979, p. 66) but these ‘‘Lindenbaum’’ constructions, for infinite languages, are not considered as effective. For finite languages, such constructions are obviously effective.

**Theorem 3.3.** *Let practical consequence operator  $C^f$  be defined on arbitrary  $L^f$ . If  $X \subset L^f$  is  $C^f$ -consistent, then there exists an effectively constructed  $Y \subset L^f$  such that  $C^f(X) \subset Y$ ,  $Y$  is  $C^f$ -consistent and  $C^f$ -complete.*

Proof. This is rather trivial for a practical consequence operator and all of the construction processes are effective. Consider an enumeration for  $L^f$  such that  $L^f = \{x_1, x_2, \dots, x_k\}$ . Let  $X \subset L^f$  be  $C^f$ -consistent and define  $X = X_0$ . We now simply construct in a completely effective manner a partial sequence of subsets of  $L^f$ . Simply consider  $X_0 \cup \{x_1\}$ . Since  $X_0$  is  $C^f$ -consistent, we have two possibilities. Effectively determine whether  $C^f(X_0 \cup \{x_1\}) = L^f$ . If so, let  $X_1 = X_0$ . On the other hand, if  $C^f(X_0 \cup \{x_1\}) \neq L^f$ , then define  $X_1 = X_0 \cup \{x_1\}$ . Repeat this construction finitely many times. (Usually, if the language is denumerable, this is expressed in an induction format.) Let  $Y = X_k$ . By definition,  $Y$  is  $C^f$ -consistent. Suppose that  $x \in L^f$ . Then there is some  $X_i$  such that either (a)  $x \in X_i$  or (b)  $C^f(X_i \cup \{x\}) = L^f$ . For (a), since  $X_i \subset Y$ ,  $x \in Y$ . For (b),  $X_i \subset Y$ , implies that  $L = C^f(X_i \cup \{x\}) \subset C^f(Y \cup \{x\}) = L^f$ . Hence,  $Y$  is  $C^f$ -complete and  $X_i \subset Y$ , for each  $i = 1, \dots, k$ . By Theorem 3.2,  $Y$  is a  $C^f$ -system. Thus  $X_0 \subset Y$  implies that  $C^f(X_0) \subset C^f(Y) = Y$ , and this completes the proof.

**Corollary 3.3.1.** *Let practical consequence operator  $C^f$  be defined on  $L^f$  and  $X \subset L^f$  be  $C^f$ -consistent. Then there exists an effectively constructed  $Y \subset L^f$  that is an extension of  $C^f(X)$  and, hence, also an extension of  $X$ , where  $Y$  is a maximally  $C^f$ -consistent  $C^f$ -system.*

Let the set  $\Sigma^p \subset \Sigma$  consist of all of science-community practical logic-systems defined on languages  $L_i^f$ . Each member of  $\Sigma^p$  corresponds to  $i \in |\Sigma^p|$  and to a practical consequence operator  $C_i^f$  defined on  $L_i^f$ . In general, the members of a set of science-community logic-systems are related by a consistency notion relative to an extended language.

**Definition 3.4.** A set of consequence operators  $\mathcal{C}$  defined on  $L$  is *system consistent* if there exists a  $Y \subset L$ ,  $Y \neq L$  and  $Y$  is a  $C$ -system for each  $C \in \mathcal{C}$ .

**Example 3.5.** Let  $\mathcal{C}$  be a set of axiomless consequence operators where each  $C \in \mathcal{C}$  is define on  $L$ . In Herrmann (2001a, b), the set of science-community consequence operators is redefined by relativization to produce a set of axiomless consequence operators, the  $S_N^V$ , each defined on the same language. Any such collection  $\mathcal{C}$  is system consistent since for each  $C \in \mathcal{C}$ ,  $C(\emptyset) = \emptyset \neq L$ .

**Example 3.6.** One of the major goals of certain science-communities is to find what is called a “grand unification theory.” This is actually a theory that will unify only the four fundamental interactions (forces). It is then claimed that this will somehow lead to a unification of all physical theories. Undoubtedly, if this type of grand unification is achieved, all other physical science theories would require some type of re-structuring. The simplest way this can be done is to use informally the logic-system expansion technique. This will lead to associated consequence operators defined on “larger” language sets.

Let a practical logic-system  $S_0$ , be defined on  $L_0^f$ , and  $L = \bigcup \{L_i^f \mid i \in \mathbb{N}\}$ ,  $\mathbb{N}$  the set of natural numbers. Let  $L_0 \subset L_1, L_0 \neq L_1$ . [Note: the remaining members of  $\{L_i^f \mid i \in \mathbb{N}\}$  need not be distinct.] Expand  $S_0$  to  $S_1 \neq S_0$  defined on  $L$  by adjoining to the logic-system  $S_0$  finitely many practical logic-system n-ary relations or finitely

many additional  $n$ -tuples to the original  $S_0$ , but where all of these additions only contain members from nonempty  $L - L_0^f$ . Although  $S_1$  need only be considered as non-trivially defined on  $L_1^f$ , if  $L \neq L_1$ , then the  $S_1$  so obtained corresponds to  $C_1$ , a consequence operator trivially extended to  $L$ . This process can be repeated in order to produce, at the least, finitely many distinct logic-systems  $S_i$ ,  $i > 1$ , that extend  $S_0$  and a set  $\mathcal{C}_1$  of distinct corresponding consequence operators  $C_i$ . Since these are science-community logic-systems, there is an  $X_0 \subset L_0^f$  that is  $C_0^f$ -consistent. By Corollary 3.3.1, there is an effectively defined set  $Y \subset L_0^f$  such that  $X_0 \subset Y$  and  $Y$  is maximally  $C_0^f$ -consistent with respect to the language  $L_0^f$ . Hence,  $C_0^f(Y) = Y \subset L_0^f$  and  $C_0^f(Y) \neq L_0^f$ . Further,  $C_0^f$  is consider trivially extended to  $L$ . Let  $Y' = Y \cup (L - L_0^f)$ . It follows that for each  $C_i$ ,  $L - L_0^f \subset C_i(L - L_0^f) \subset L - L_0^f \neq L$ . By construction, for each  $C_i$ ,  $C_i(Y) = Y$ ; and for each  $X \subset L$ ,  $C_i(X) = C_0(X \cap L_0^f) \cup C_i(X \cap (L - L_0^f))$ . So, let  $X = Y'$ . Then for each  $C_i$ ,  $C_i(Y') = C_0(Y) \cup (L - L_0^f) = Y \cup (L - L_0^f) = Y' \neq L$ . Hence, the set of all  $C_i$  is system consistent.

**Example 3.7.** Consider a denumerable language  $L$  and Example 3.2 in Herrmann (1987). [Note: There is a typographical error in this 1987 example. The expression  $x \notin \mathcal{U}$  should read  $x \notin U$ .] Let  $\mathcal{U}$  be a free-ultrafilter on  $L$  and let  $x \in L$ . Then there exists some  $U \in \mathcal{U}$  such that  $x \notin U$  since  $\bigcap \mathcal{U} = \emptyset$  and  $\emptyset \notin \mathcal{U}$ . Let  $B = \{x\}$  and  $\mathcal{C} = \{P(U, B) \mid U \in \mathcal{U}\}$ , where  $P(U, B)$  is the finitary consequence operator defined by  $P(U, B)(X) = U \cup X$ , if  $x \in X$ ; and  $P(U, B)(X) = X$ , if  $x \notin X$ . [Note: this is the same operator  $P$  that appears in the proof of Theorem 6.4 in Herrmann (2001a, b).] There, at the least, exists a sequence  $S = \{U_i \mid i \in \mathbb{N}\}$  such that  $U_0 = U$  and  $U_{i+1} \subset U_i$ ,  $U_{i+1} \neq U_i$ . It follows immediately from the definition that  $P(U_{i+1}, B) \leq P(U_i, B)$  and  $P(U_{i+1}, B)(B) = U_{i+1} \cup B \subset U_i \cup B$ , for each  $i \in \mathbb{N}$ . Hence, in general,  $P(U_{i+1}, B) < P(U_i, B)$  for each  $i \in \mathbb{N}$ . Let  $Y = L - \{x\}$ . Then  $P(U_i, B)(Y) = U_i \cup (L - \{x\}) = L - \{x\} = Y$ ,  $i \in \mathbb{N}$ . Thus, the collection  $\{P(U_i, B) \mid i \in \mathbb{N}\}$  is system consistent.

**Theorem 3.8.** Consider  $\mathcal{A} \subset \mathcal{C}_f$  defined on  $L$  and the  $(\leq)$  least upper bound  $\bigvee_w \mathcal{A}$ . Then  $\bigvee_w \mathcal{A} \in \mathcal{C}_f$  and if  $\mathcal{A}$  is system consistent, then there exists some  $Y \subset L$  such that  $Y = \bigvee_w \mathcal{A}(Y) = C(Y) \neq L$  for each  $C \in \mathcal{A}$  and  $\bigvee_w \mathcal{A} \neq U$ . Further, if  $X \subset L$ ,  $X \neq L$ , is a  $C$ -system for each  $C \in \mathcal{A}$ , then  $X = \bigvee_w \mathcal{A}(X) = C(X) \neq L$  for each  $C \in \mathcal{A}$ .

Proof. Corollary 2.10.1 yields the first conclusion. From the definition of system consistent, there exists some  $Y \subset L$  such that  $C(Y) = Y \neq L$  for each  $C \in \mathcal{A}$ . From Lemma 2.6, for each  $C \in \mathcal{A}$ ,  $\bigvee_w \mathcal{A}(Y) = C(Y) \neq L$ . Hence,  $\bigvee_w \mathcal{A} \neq U$ . The last part of this theorem follows from Lemma 2.6 and the fact that  $X$  is also a  $\bigvee_w \mathcal{A}$ -system. This completes the proof.

#### 4. Applications

In Herrmann (2001a,b), the relativized set  $\{S_{N_i}^V(X) \mid i \in \mathbb{N}\}$ , when  $|\{S_{N_i}^V(X) \mid i \in \mathbb{N}\}| = \aleph_0$ , is introduced. This set is system consistent and is unified through application of Theorem 3.8. Assuming system consistency, this also applies to the unrelativized case where each relativized consequence operator  $S_{N_i}^V$  is replaced with

the physical theory consequence operator  $S_{N_i}$ . Also note that  $S_{N_i}$  and  $S_{N_i}^V$  are usually considered practical consequence operators.

Depending upon the set  $\mathcal{C}$  of consequence operators employed, there are usually many  $X \subset L$ ,  $X \neq L$  such that  $X$  is a  $C$ -system for each  $C \in \mathcal{C}$ . For example, we assumed in Herrmann (2001a, b) that there are two 1-ary relations for the science-community logic-systems. One of these contains the logical axioms and the other contains a set of physical axioms; a set of natural laws. Let  $\{S'_{N_i} \mid i \in \mathbb{N}\}$  be the set of science-community corresponding consequence operators relativized so as to remove the set of logical theorems. Each member of a properly stated set of natural laws,  $N_j$ , used to generate the consequence operators  $\{S'_{N_i} \mid i \in \mathbb{N}\}$  should be a  $C$ -system for each member of  $\{S'_{N_i} \mid i \in \mathbb{N}\}$ . As mentioned, the physical theories being considered here are not theories that produce new “natural laws.” The argument that the Einstein-Hilbert equations characterize gravitation fields, in general, leads to the acceptance by many science-communities of these equations as a “natural law” that is then applied to actual physical objects. Newton’s Second Law of motion is a statement about the notion of inertia within our universe. It can now be derived from basic laboratory observation and has been shown to hold for other physical models distinct from its standard usage (Herrmann, 1998). The logic-systems that generate the members of  $\{S'_{N_i} \mid i \in \mathbb{N}\}$  have as a 1-ary relation a set of natural laws. Then one takes a set of specific physical hypotheses  $X$  that describes the behavior of a natural-system and applies the logic-system to  $X$ . This gives a statement as to how these natural laws affect, if at all, the behavior being described by  $X$ . It is this approach that implies that each properly described  $N_j \neq L$  is a  $C$ -system for each  $C \in \{S_{N_i} \mid i \in \mathbb{N}\}$ . Hence, Theorem 3.8 applies to  $\mathcal{C} = \{S'_{N_i} \mid i \in \mathbb{N}\}$ .

At any moment in human history, one can assume, due to the parameters present, that there is, at the least, a denumerable set of science-community logic-systems or that there exist only a finite collection of practical logic-systems defined on finite  $L^f$ . The corresponding set  $\mathcal{C}^f = \{C_i^f \mid i = 1, \dots, n\} \subset \mathcal{C}_f^f$  of practical consequence operators would tend to vary in cardinality at different moments in human history. For the corresponding finite set of practical consequence operators, by Theorem 2.10, there is a standard (least upper bound) practical consequence operator  $\mathcal{U}$ , and hence “the best” practical logic-system, that unifies such a finite set. The following result is a restatement of Theorem 3.8 for such a finite set of practical consequence operators.

**Theorem 4.1.** *Let  $L^f$  and  $\mathcal{C}^f$  be defined as above. Suppose that  $\mathcal{C}^f$  is system consistent.*

(i) *Then there exists a practical consequence operator  $\mathcal{U}_1 \in \mathcal{C}_f^f$  defined on the set of all subsets of  $L^f$  such that  $\mathcal{U}_1 \neq \mathcal{U}$ , and a  $W \subset L$  such that, for each  $C_i^f \in \mathcal{C}^f$ ,  $C_i^f(W) = \mathcal{U}_1(W) = W \neq L^f$ , where  $\mathcal{U}_1(W) \subset L^f$ .*

(ii) *For each  $X \subset L^f$ ,  $\bigcup\{C_i^f(X) \mid i = 1, \dots, n\} \subset \mathcal{U}_1(X) \subset L^f$  and  $\mathcal{U}_1$  is the least upper bound in  $\langle \mathcal{C}_f^f, \vee_w, \wedge, I, U \rangle$  for  $\mathcal{C}^f$ .*

(iii) *Let  $X \subset L^f$  and  $X \neq L^f$  be a  $C_i^f$ -system for each  $C_i^f \in \mathcal{C}^f$ . Then  $X = C_i^f(X) = \mathcal{U}_1(X) \neq L^f$ , for each  $i = 1, \dots, n$ .*

Letting finite  $\mathcal{C}^f$  contain practical consequence operators either of the type  $S_{N_i}$ ,  $S_{N_i}^V$  or  $S_{N_i}'$ , exclusively, then  $\mathcal{U}_1$  would have the appropriate additional properties and would generate a practical logic-system. Corollary 2.10.1 and Theorem 3.8 yield a more general unification  $\bigvee_{\mathbf{w}} \mathcal{A}$ ,  $\mathcal{A} \subset \mathcal{C}_f$ , as represented by a least upper bound in  $\langle \mathcal{C}_f, \bigvee_{\mathbf{w}}, \wedge, I, U \rangle$ , with the same properties as stated in Theorem 4.1. Thus depending upon how physical theories are presented and assuming system consistency, there are nontrivial standard unifications for such physical theories. Further, system consistency is used only so that one statement in Theorem 3.8, Theorem 4.1 and this paragraph will hold. This one fact is that each of the standard unifications of a collection  $\mathcal{A} \subset \mathcal{C}_f$  is not the same as the upper unit if and only if the  $\mathcal{A}$  is system consistent. Further, if an  $X \subset L^f$  [resp.  $X \subset L$ ] is  $\mathcal{U}_1$ -consistent [resp.  $\bigvee_{\mathbf{w}} \mathcal{A}$ -consistent], then  $X$  is  $C$ -consistent for each  $C \in \mathcal{C}$  [resp.  $C \in \mathcal{A}$ ].

For General Intelligent Design Theory, the unification  $\bigvee_{\mathbf{w}} \mathcal{A}$  can be considered as a restriction of the ultralogic  $^* \bigvee_{\mathbf{w}} \mathcal{A}$  and can, obviously, be interpreted as an intelligence that designs and controls the combined behavior exhibited by members of  $\mathcal{C} = \{S_{N_i}' \mid i \in \mathbb{N}\}$ , as they are simultaneously applied to a natural-system.

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